
TOPICS IN ALGEBRA

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By

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FOREWORD

This book has been prepared by the authors to meet the requirements of the Freshman Mathematics courses given at the Worcester Polytechnic Institute. It is their opinion that this selection of topics from the field of College Algebra comprises those that are most essential in the training of students for the engineering profession.

Within the book itself, Doctor Morley is responsible for the preparation of the material in Chapter IV, and Professor Gay for that in Chapters I, II, and III.

The authors wish to express their thanks to their colleagues, Professor Harris Rice, Professor William L. Phinney, Jr., Professor Edward C. Brown, and Mr. David Kiley for their generous assistance in checking the text and the answers.

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TOPICS IN ALGEBRA

Chapter I

PERMUTATIONS, COMBINATIONS, AND PROBABILITY

I-1. INTRODUCTION. The meaning of "permutation" and "combination" may be explained best by considering them simultaneously. Suppose we have a group of objects; to be concrete, five flags of different colors, red, blue, green, yellow, and black. If we select three of them, say the red, blue, and green, paying no attention to the order of selections, or of their arrangement with respect to each other in any way, it is said that we have taken a "combination of the five flags three at a time." Of course, there are many other possible combinations of the five flags three at a time.

Next suppose that we take three of the five flags and display them in a row for a signal, each different order of the flags in the row being a different signal according to a certain code. Each such possible order of the flags is called a "permutation of the five flags three at a time."

These examples are in accord with the following definitions.

Definition. An arrangement of a given number of elements is called a permutation.

Definition. Any one of the sets into which a number of elements may be grouped, no account being taken of the possible permutations of the set, is called a combination.

In problems involving permutations and combinations the object is to find the total number possible. In the following sections we shall discuss formulas and methods for doing this.

The study of probability is closely allied to permutations and combinations; many problems in probability are based on them so it is appropriate to group all three in one chapter leaving the discussion of probability for the last.

I. PERMUTATIONS

I-2. PERMUTATIONS OF n ELEMENTS TAKEN r AT A TIME. We state the following

Fundamental Principle. If an act which can be performed in any of a ways is followed by an act which can be performed in any of b ways, the number of ways in which the two acts may be performed in succession is the product ab .

Proof. Let the first act be performed in some one way, and let this performance be followed by the performance of the second act in all the b possible ways; let the first act be performed in a different way, and let this performance be followed by all the b possible ways of doing the second act; let this be repeated for each of the a possible ways of doing the first act. Then the total number of ways in which the two acts may be performed in succession is $\underbrace{(b + b + b + \dots)}_{a \text{ times}} = ab$ ways.

The principle may be extended to any number of factors.

Returning to the flags in §I-1, how many permutations of the five flags three at a time are possible? If we think of the flags as being displayed in a row, we have five choices for the first position, four choices for the second position, and three choices for the third. Applying the Fundamental Principle, the number of permutations of the flags three at a time is

$$5 \cdot 4 \cdot 3 = 60 \text{ permutations.}$$

This example suggests the

Theorem. The permutations of n different elements taken r at a time is given by the formula

$${}_nP_r = n(n-1)(n-2)\dots(n-r+1). \quad [\text{I-I}]$$

Proof. The first act in making the permutation may be performed in n ways; the second in $n-1$ ways; the r th act, which completes the permutation, in $n-r+1$ ways. Applying the Fundamental Principle we obtain the formula stated.

The notation ${}_nP_r$ is read, "the permutations of n things taken r at a time."

We see at once from [I-I] that the permutations of n things taken all at a time is

$${}_nP_n = n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1 = n!. \quad [\text{I-II}]$$

The symbol $n!$ is called factorial n . (A different notation often employed for factorial n is $[n]$.)

Note. By definition, $0! = 1$.

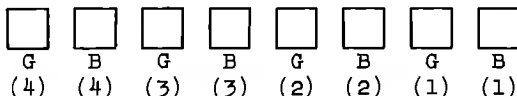
Example 1. In how many ways may 4 girls and 4 boys be seated in a row of 8 chairs?

Solution. This problem requires the number of permutations of 8 elements taken all at a time. By [I-II],

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320 \text{ ways.}$$

Example 2. In how many ways may 4 girls and 4 boys be seated in a row of 8 chairs if they are seated alternately with a girl seated in the chair at the left end of the row?

Solution. Occasionally a diagram is useful in such problems.



The diagram shows the arrangement, the figures indicating the number of persons available for each chair. Applying the Fundamental Principle, the product of these numbers gives 576 ways.

The same example may also be solved by formulas. The girls may be seated in the 4 seats reserved for them in $4! = 24$ ways. Similarly, the boys may be seated in $4!$ ways. By the Fundamental Principle they may all be seated in

$$4! \cdot 4! = (24)(24) = 576 \text{ ways.}$$

Example 3. In how many ways may 4 girls and 4 boys be seated alternately in a row of 8 chairs?

Solution. This example differs from Example 2 by allowing either a boy or a girl to sit in the chair at the left end of the row.

By Example 2, starting with a girl at the left, we have 576 ways. Similarly, starting with a boy at the left, we have 576 ways. Adding, we have 1152 ways.

Note that in the final operation we added the two items. Some students at first make the mistake of multiplying in such situations, but this is not a case for applying the Fundamental Principle. We are here dealing with the possibility of seating either a girl at the left, or a boy at the left, and the sum represents the total number of ways of doing one thing or the other. By contrast, multiplying the numbers would give the number of ways in which one group of eight could be combined with a second group of eight, involving a total of sixteen persons.

I-3. SPECIAL TYPES OF PERMUTATIONS.

1. Permutations with repetition allowed. In the preceding text and the examples no repetition was contemplated; e.g., two persons were not allowed to occupy the same chair. In some problems it is natural to allow repetition.

Example 1. In how many ways may 4 persons enter a building having 3 doors?

Solution. The first person may enter in any of three ways; the second, third, and fourth have, presumably, the same privilege. By the Fundamental Principle all may enter in $3 \cdot 3 \cdot 3 \cdot 3 = 81$ ways.

A similar argument using the letters n and r leads to the

Theorem. If each of r elements may be permuted in n ways, repetition being allowed, the total number of permutations is

$$P = n^r. \quad [\text{I-III}]$$

2. Permutations of Elements Some of which Are Identical.
Assume that we have 5 balls, 2 red and 3 white. If they are all different, they may be permuted in $5! = 120$ ways. If, however, the 2 red balls are regarded as being identical, so that interchanging them with each other does not constitute an essentially different arrangement, it is evident that the total number of arrangements must be divided by two. Similarly, if the 3 white balls are regarded as identical, the total number of ways must be divided by the ways the 3 white balls may be permuted among themselves, viz., by $3!$. Hence, the number of essentially different permutations of the 5 balls is $\frac{5!}{2! \cdot 3!} = 10$ ways. This is an illustration of the

Theorem. If n_1 elements of a group are identical, n_2 others are identical, and so on for m sub-groups, the permutations of the group taken all at a time are

$$P = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_m!} \quad [\text{I-IV}]$$

where n is the total number of elements.

Note that in [I-IV] the elements are taken all at a time. We shall not consider any cases where fewer are taken.

3. Circular Permutations. In some problems the elements may be arranged in a circle instead of in a row. For example, suppose 8 persons are seated around a circular table. If each person moves to his right the same number of chairs, they are still in essentially the same order. To find how many essentially different permutations are possible, think of one person as taking a fixed position. The remaining seven may be seated in $7!$ ways. This is an illustration of the

Theorem. The permutations of n things in a circle taken all at a time are

$$P = (n - 1)!. \quad [\text{I-V}]$$

EXERCISE I-1

1. Compute $2!$, $3!$, $4!$, and so on up to $10!$.
2. Evaluate (a) ${}_6P_3$; (b) ${}_8P_4$; (c) ${}_{17}P_5$.
3. Evaluate (a) ${}_7P_2$; (b) ${}_{11}P_3$; (c) ${}_{15}P_7$.
4. A schooner has 5 sails, flying jib, jib, stays'l, fores'l, and mains'l. In how many orders may they all be hoisted if the jib must be hoisted first?

5. A lady on vacation has 10 dinner dresses. If she dresses for dinner on 6 evenings, in how many orders may she display her dresses, no dress being worn twice?

6. A General has acquired 15 medals. In how many ways may he arrange 5 in a row on his breast?

7. The manager of a basketball team has 5 players who can play either of the two forward positions, 3 centers, and 4 who can play either of the two guard positions. In how many ways may he arrange his line-up?

8. (a) Write all the permutations of Q, R, and S taken two at a time; taken all at a time. (b) Do the same as in (a) for Q, R, S, and T.

9. In how many ways may 6 different coins be distributed among 4 boys if each boy may receive any number of coins?

10. (a) 3 tourists go to a resort where there are 8 hotels. In how many ways may they select places to stay? (b) 8 tourists go to a resort where there are 3 hotels. In how many ways may they select places to stay?

11. In how many ways may 7 persons be seated in a row if 2 of them must be seated side by side?

12. (a) The same as Problem 11 if 3 must be seated side by side? (b) If 4 must be seated side by side?

13. (a) In how many ways may 7 persons be seated in a row if 2 of them must not be seated side by side? (b) If 3 of them must not be seated side by side?

14. In how many ways may 10 vases of flowers be arranged in a row if 2 of them must not be side by side?

15. (a) If two coins are tossed, in how many ways may they fall? (b) 3 coins? (c) 5 coins? (d) n coins?

16. (a) If 2 dice are tossed, in how many ways may they fall? (b) 3 dice? (c) 5 dice? (d) n dice?

17. An exhibitor at a flower show has to display 5 varieties of roses and 4 of peonies in a row. In how many ways may he do so if all the roses must be together, and all the peonies must be together?

18. The same as Problem 17 if only the roses must be together.

19. In how many ways may 5 identical algebra books and 3 other books be arranged in a row of 8 on a shelf?

20. In how many ways may 3 identical algebras, 5 identical grammars, and 4 identical lexicons be arranged in a row of 12?

21. How many essentially different arrangements may be made using all the letters in the word arrangement?

22. How many essentially different arrangements may be made using all the letters in the word possibilities?

23. Containers for salt, pepper, vinegar, mustard, olive oil, catsup, and worcestershire sauce are arranged in a circle in the center of a table. How many essentially different orders are possible?

24. A man and his wife invite 6 persons to dinner. In how many essentially different ways may they all sit around a circular table (a) if any person may have any seat? (b) If the hostess must occupy a certain chair? (c) If the host and hostess must occupy certain chairs?

25. Given the digits 2, 4, 6, 7, and 8, how many different positive, whole numbers may be formed (a) containing 3 digits, no digit occurring twice in the same number? (b) Containing 4 digits, no digit occurring twice in the same number? (c) Containing either 3 or 4 digits as in (a) and (b)?

26. Using the data in Problem 25, answer (a), (b), and (c) if repetition of the digits is allowed?

27. Using 7 different flags how many signals may be formed consisting of any number of flags displayed in a row?

28. Using 3 identical red flags, 4 identical blue flags, and a white flag, how many signals may be formed by displaying them all in a line?

29. When a basketball player shoots a foul the 9 other players line up along both sides of the shooting zone, 4 on one side and 5 on the other, the members of the two teams, including the shooter, alternating around the circuit. In how many ways may they arrange themselves?

30. From a party of 8 persons, in how many essentially different ways may 5 be seated around a table to play rummy?

31. From a group of 8 keys, in how many essentially different ways may 5 be arranged on a key ring?

II. COMBINATIONS

I-4. THE COMBINATIONS OF n ELEMENTS TAKEN r AT A TIME. The distinction between the terms permutation and combination was made in §I-1 which may be reviewed at this point. Since then we have derived the formula for ${}_nP_r$ which gives the number of permutations of n things taken r at a time. We seek a corresponding formula for the combinations of n things taken r at a time.

Concretely, suppose we have five different flags; red, blue, green, yellow, and black. We know that the number of permutations of these flags three at a time is given by

$${}_5P_3 = 5 \cdot 4 \cdot 3 = 60.$$

In considering combinations, by definition, we ignore the permutations of each possible set of three. Thus, red, blue, green; red, green, blue; blue, red, green; etc., are all different permutations, but one and the same combination. Since any set of three elements may be permuted in $3!$ ways, the 60 permutations must be divided by $3!$. This reduces the 60, the number of permutations, to 10, the number of combinations. A similar argument expressed in terms of the letters n and r leads to the

Theorem. The combinations of n different elements taken r at a time equals the number of permutations of the n elements taken r at a time divided by $r!$.

${}_nC_r$ is the notation employed for "the combinations of n elements taken r at a time." Hence, the theorem may be expressed as the formula

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} \quad [\text{I-VI}]$$

Formula [I-VI] may also be written

$$(1) \quad ({}_nC_r)(r!) = {}nP_r.$$

This form expresses the point of view opposite to the one outlined above. It says that if we take all the possible combinations of n elements r at a time (even though their number is unknown) and multiply each one by all the permutations possible in its group, i.e., by $r!$, we are bound to obtain all the possible permutations of n elements taken r at a time. Since ${}_nP_r$ and $r!$ are known, we may solve (1) for the unknown term ${}_nC_r$.

Example 1. Find the combinations of 11 marbles taken 4 at a time.

Solution. By Formula [I-VI], ${}_{11}C_4 = \frac{11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} = 330$ combinations.

Note that there will always be the same number of factors in both numerator and denominator.

Example 2. Evaluate ${}_{16}C_3$ and ${}_{16}C_{13}$.

Solution. ${}_{16}C_3 = \frac{16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3} = 560.$

$${}_{16}C_{13} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13} = 560.$$

Note that after cancelling like factors in the second part, both parts reduce to the same set of factors. This is an example of the short cut in computation afforded by the formula

$${}_nC_r = {}_nC_{n-r}. \quad [\text{I-VII}]$$

The logic of this formula is obvious if we consider that for each set of r elements taken from a group of n elements, there is automatically formed a group of $(n-r)$ omitted elements.

Example 3. A man has 15 friends whom he would like to invite to a party, but he has accommodations for only 12. In how many ways may he choose his guests?

Solution. A party of 12 persons may be selected from 15 persons in ${}_{15}C_{12}$ ways. To reduce the labor of computation apply Formula [I-VII], or,

$${}_{15}C_{12} = {}_{15}C_3 = \frac{15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3} = 455 \text{ ways.}$$

Example 4. In how many ways may a joint committee of 3 Freshmen and 4 Sophomores be chosen from 5 Freshmen and 7 Sophomores?

Solution. 3 Freshmen may be selected from 5 in ${}_5C_3 = 10$ ways. 4 Sophomores may be selected from 7 in ${}_7C_4 = 35$ ways. Applying the Fundamental Principle, the entire committee may consist of any of the 10 groups of Freshmen joined with any of the 35 groups of Sophomores in $10 \cdot 35$ ways, or 350 ways.

Example 5. In how many ways may a committee of 7 containing at least 4 Sophomores be chosen from the Freshmen and Sophomores in Example 4?

Solution. A committee of at least 4 Sophomores allows 4 Sophomores and 3 Freshmen, 5 Sophomores and 2 Freshmen, 6 Sophomores and 1 Freshman, or 7 Sophomores. Using the result of Example 4,

4 Sophomores and 3 Freshmen: ${}_7C_4 \cdot {}_5C_3 = 35 \cdot 10 = 350$ ways.
Similarly,

5 Sophomores and 2 Freshmen: ${}_7C_5 \cdot {}_5C_2 = 21 \cdot 10 = 210$ ways.

6 Sophomores and 1 Freshman: ${}_7C_6 \cdot {}_5C_1 = 7 \cdot 5 = 35$ ways.

7 Sophomores and no Freshmen: ${}_7C_7 \cdot {}_5C_0 = 1 \cdot 1 = 1$ way.
596 ways.

The sum of the possibilities listed above is the total number of ways of forming the committee.

I-5. RELATION BETWEEN THE BINOMIAL THEOREM AND THE FORMULAS FOR COMBINATIONS. Example 5 in the preceding section suggests the question, what is the total number of combinations of n elements taken one at a time, then two at a time, and so on,

up to n at a time. The answer to this question may be found with the help of the binomial theorem. Expanding $(1+x)^n$ we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \dots + x^n.$$

The coefficient of the second term is ${}_nC_1$; of the third, ${}_nC_2$; and so on, and of the last ${}_nC_n$. Hence, we may rewrite the expansion as

$$(1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + {}_nC_3x^3 + \dots + {}_nC_nx^n.$$

Now let $x = 1$, transpose the 1 in the right member to the left, and

$$2^n - 1 = {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n. \quad [\text{I-VIII}]$$

Stated in words, the total number of combinations of n elements taken one at a time, two at a time, and so on up to n at a time is given by the expression $2^n - 1$.

Example 1. What is the total number of combinations of 8 soldiers taken 1 at a time, 2 at a time, and so on up to all at a time?

Solution. Applying Formula [I-VIII], the answer is $2^8 - 1 = 256 - 1 = 255$ combinations. It is suggested that the student verify this answer by working out the separate combinations and adding them.

EXERCISE I-2

1. Evaluate (a) ${}_7C_3$; (b) ${}_{20}C_2$; (c) ${}_{50}C_{47}$; (d) ${}_8C_1$; (e) ${}_8C_8$; (f) ${}_8C_0$.

2. Evaluate (a) ${}_9C_4$; (b) ${}_{17}C_3$; (c) ${}_{100}C_{98}$; (d) ${}_{10}C_1$; (e) ${}_{10}C_{10}$; (f) ${}_{10}C_0$.

3. At a party where 12 persons were present each shook hands with everyone else. How many handshakings were there?

4. How many straight lines can be drawn through 9 points no three of which are in a straight line?

5. In how many ways may a group of three be chosen from a class of twenty to work at the board?

6. If a class consists of 8 girls and 12 boys, in how many ways may a group of 5 consisting of 2 girls and 3 boys be chosen to work at the board?

7. Referring to the class in Problem 6, in how many ways may the group be chosen if it is to contain at most 3 boys?

8. Referring to the class in Problem 6, in how many ways may the group of 5 be chosen if it is to contain at least 3 boys?

9. From a dollar, a half dollar, a quarter, a dime, and a nickel, how many sums of money can be made using (a) just 3 coins? (b) Using either 3 or 4 coins? (c) Using any number of the coins?
10. Using the coins mentioned in Problem 9, how many sums of money exceeding a quarter may be formed?
11. If 4 coins are tossed, in how many ways may they fall 1 head and 3 tails?
12. If 4 coins are tossed, in how many ways may they fall (a) 4 heads? (b) Exactly 3 heads? (c) Exactly 2 heads? (d) Exactly 1 head? (e) No heads? Can you check the answers to this problem with the result of Problem 15(d) in Exercise I-1?

The following problems may involve both permutations and combinations.

13. A lady has 6 vases of larkspur and 5 of roses; in how many ways may she display 3 vases of larkspur and 2 of roses in a row if each possible order of the vases counts as a different display?
14. At a picnic the manager of the Fat-men's baseball team has 2 catchers, 3 pitchers, a fixed infield of 4 ex-big-leaguers who always play the same positions, and 5 outfielders who can play any outfield position. In how many ways may he field his team?
15. In how many ways may the manager in Problem 14 arrange a batting order if a pitcher always bats last, and a catcher next to last?
16. A swing orchestra of seven has one drummer, one pianist, and 5 others; 3 of the 5 can play either saxophone or trombone, the fourth can play either cornet or bass viol, and the fifth either piccolo or clarinet. How many different combinations of instruments can be formed?
17. In how many ways may 9 different problems be all assigned to 3 students, each student to receive 3 problems?
18. A party of 12 guests are met at a station by a beach wagon, a sedan, and a roadster. If the beach wagon accommodates 6, the sedan 4, and the roadster 2, in how many ways may they all be given transportation?
- A pack of cards consists of 52 cards divided evenly among 4 suits (spades, hearts, diamonds, clubs), each suit containing ace, king, queen, jack, 10, 9, 3, 2.
19. How many different hands of 5 cards may be dealt from a pack?

20. How many different hands of 5 cards may be dealt from a pack, each hand containing 3 aces and 2 other cards of the same rank (kind)?

21. How many different hands of 5 cards may be dealt from a pack (a) all of the heart suit? (b) All of the same suit?

22. How many different hands of 5 cards may be dealt from a pack all of the same suit and in sequence?

23. From a pack of cards in how many ways may 7 cards of the same suit be arranged in a row on a table?

24. From a pack of cards in how many ways may 4 hands of 13 cards each be dealt and distributed to 4 players? (Leave answer in factored form.)

25. From a group of 12 marines in how many ways may (a) a squad of 8 be chosen? (b) A squad of 8 be chosen and arranged in two ranks of 4 men each?

26. A man has 10 friends, among them X and Z who are not on speaking terms. In how many ways may he invite 6 of them to a party if both X and Z may not be included?

27. There are 5 candidates for a society. In how many ways may groups of one or more of them be summoned before the society?

28. How many different groups of one or more actors may appear on the stage at once if there are 10 actors in the cast?

29. An examination paper contains 10 questions of which nos. 1, 2, and 3 must be answered. In how many ways may a student choose his questions if he must answer (a) just 7 questions? (b) At least 7 questions?

30. In a twilight baseball league there are 10 clubs each of which plays 7 games with every other club. How many games must be scheduled for the whole league?

31. If a state makes automobile registration plates which contain just two letters of the alphabet, excluding I and O, followed by numbers of one, two, or three digits, of which the first digit may not be zero, how many automobiles may be registered?

III. PROBABILITY

I-6. PROBABILITY. The words "probable" and "probability" are used rather loosely in everyday life. In what follows we shall consider first mathematical probability, and second, experimental, or empirical, probability.

If s denotes the number of ways in which it is equally likely that an event may occur, and f denotes the number of ways

in which it is equally likely that the event may fail to occur, then the mathematical probability that it will occur is

$$p = \frac{s}{s + f} . \quad [\text{I-IX}]$$

The value of p may range from 0 to 1. When $s = 0$, $p = 0$, which means that the event certainly will not happen; if $f = 0$, $p = 1$, which represents certainty that the event will happen.

We have chosen to emphasize the probability of occurrence of an event; in the same notation the probability that the event will not occur is given by

$$q = \frac{f}{s + f} .$$

Notice especially the useful relation,

$$p + q = 1 . \quad [\text{I-X}]$$

Example 1. If a die is tossed, what is the probability that a 5 will be thrown?

Solution. Assuming that it is equally likely that any one of the six faces of the die may appear on top, the probability that the face numbered 5 will appear is $\frac{1}{6}$.

Note 1. Although the concept that all the possibilities of success and failure are equally likely is of fundamental importance in the theory of probability, it will be assumed that such is the case in the remainder of the chapter without repeating the words in each instance. In the same vein when we speak of tossing dice, drawing balls from a bag, etc., it will be assumed that the act is performed at random.

Note 2. To avoid repetition, unless the contrary is indicated, the phrase "probability of an event" will be understood to mean the "probability of occurrence of the event."

I-7. PROBABILITY OF MUTUALLY EXCLUSIVE EVENTS. Several events are said to be mutually exclusive if the occurrence of any one of them renders impossible the occurrence of any of the others.

Theorem. If several events are mutually exclusive, the probability of one or other is the sum of the probabilities of the separate events.

Proof. Let s_1 denote the number of ways in which an event e_1 may occur; let s_2 denote the same for event e_2 , and so on, and let e_1, e_2, \dots be mutually exclusive. Let t denote the total number of ways in which e_1, e_2, \dots may occur and fail to occur. By definition, the probability that the first or second will occur is

$$p = \frac{s_1 + s_2}{t} = \frac{s_1}{t} + \frac{s_2}{t} .$$

But $\frac{s_1}{t}$ and $\frac{s_2}{t}$ are the probabilities of the first and second events, respectively. Hence, p is the sum of their respective probabilities. The proof is readily extended to as many events as desired, say m . If we denote $\frac{s_1}{t}$ by p_1 , $\frac{s_2}{t}$ by p_2 , $\frac{s_m}{t}$ by p_m , we may write the formula for the probability of one or another of m mutually exclusive events as

$$p = p_1 + p_2 + \dots + p_m. \quad [\text{I-XI}]$$

Example. What is the probability of tossing a 2, or a 3, or a 5 in a single toss of a die?

Solution. Here $s_1 = s_2 = s_3 = 1$, and $t = 6$ ways; also, $p_1 = p_2 = p_3 = \frac{1}{6}$. Hence, $p = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

I-8. PROBABILITY OF INDEPENDENT AND DEPENDENT EVENTS.

Two, or more events are said to be independent if the occurrence of one does not affect the probability of any of the others; they are said to be dependent if the occurrence of one does affect the probability of others.

Theorem. The probability of several independent events is equal to the product of their several probabilities.

Proof. Consider two independent events, e_1 and e_2 . Let s_1 be the number of ways in which e_1 may occur, and t_1 the number of ways in which it may occur and fail to occur; let s_2 and t_2 have similar meanings for e_2 . By the Fundamental Principle $(t_1)(t_2)$ is the number of ways in which both may occur and fail to occur, and $(s_1)(s_2)$ is the number of ways in which both may occur. By definition, the probability that both will occur is

$$p = \frac{(s_1)(s_2)}{(t_1)(t_2)} = \left(\frac{s_1}{t_1}\right)\left(\frac{s_2}{t_2}\right).$$

But, by definition, $\frac{s_1}{t_1}$ and $\frac{s_2}{t_2}$ are the probabilities of e_1 and e_2 , respectively. This establishes the theorem for two events, and it may be extended similarly for any number. Summarized as a formula, the probability of all of n independent events whose separate probabilities are p_1, p_2, \dots, p_n is

$$p = p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_n. \quad [\text{I-XII}]$$

Example 1. What is the probability of throwing a tail and a 5 by tossing a coin and a die?

Solution. The probability of throwing a tail is $\frac{1}{2}$; of throwing a 5, $\frac{1}{6}$. The events are obviously independent. Applying [I-XII] the probability of throwing both is $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$.

We consider next the case of dependent events.

Theorem. If p_1 is the probability of an event e_1 , and if, after e_1 has occurred, p_2 is the probability of event e_2 , then the probability that both will occur in succession is the product of their probabilities.

Proof. Let s_1 be the number of ways in which an event e_1 may occur, and let t_1 be the number of ways in which it may occur and fail to occur. Assuming that e_1 occurs, let s_2 and t_2 have similar meanings for a second event e_2 . By the Fundamental Principle the two events may occur in succession in $(s_1)(s_2)$ ways, and may occur or fail to occur in $(t_1)(t_2)$ ways. By definition, the probability that the events will occur in succession is

$$p = \frac{(s_1)(s_2)}{(t_1)(t_2)} = \left(\frac{s_1}{t_1}\right) \cdot \left(\frac{s_2}{t_2}\right) = (p_1)(p_2), \quad [\text{I-XIII}]$$

where p_1 and p_2 have their usual meanings.

The proof may be extended for any number of dependent events performed in succession.

Example 2. A bag contains 3 white and 7 red balls. If 2 are drawn in succession (the first being not replaced), what is the probability that both are white?

First Solution. The probability of drawing a white ball on the first draw is $\frac{3}{10}$. If the draw is successful, there are 2 white and 7 red balls remaining in the bag, and the probability of drawing a white ball on the second draw is $\frac{2}{9}$. By the preceding theorem the probability of drawing two white balls in succession is

$$p = \frac{3}{10} \cdot \frac{2}{9} = \frac{6}{90} = \frac{1}{15}.$$

Second Solution. A second and very useful way of approaching this problem is the following: 2 balls can be taken from 10 balls in ${}_{10}C_2$ ways; 2 white balls can be taken from 3 white balls in ${}_3C_2$ ways. The probability of drawing 2 white balls from the bag of 10 is

$$p = \frac{{}_3C_2}{{}_{10}C_2} = \frac{3}{45} = \frac{1}{15}.$$

The second solution corresponds to the term $\frac{(s_1)(s_2)}{(t_1)(t_2)}$ in formula [I-XIII], while the first solution corresponds to

$\left(\frac{s_1}{t_1}\right) \cdot \left(\frac{s_2}{t_2}\right)$. Taken together they show that it is immaterial whether the two balls are taken in succession (without replacing the first), or whether they are taken simultaneously at a single draw.

Example 3. A bag contains 5 white and 8 red balls. If 3 are drawn, what is the probability that 2 are white and 1 is red?

Solution. 3 balls may be drawn from 13 in ${}_{13}C_3$ ways. The desired group of 2 white and 1 red may occur in ${}_5C_2 \cdot {}_8C_1$ ways. The probability that one of these groups will be drawn is

$$p = \frac{{}_5C_2 \cdot {}_8C_1}{{}_{13}C_3} = \frac{10 \cdot 8}{286} = \frac{40}{143}.$$

Here a solution modeled on the second solution of Example 2 is preferable to one modeled on the first.

Some problems involve a combination of several of the cases which we have been considering separately.

Example 4. One bag contains 5 white and 3 red balls; another bag contains 6 white and 4 black balls. If a ball is drawn from each bag, what is the probability that at least one is white?

Solution. The probability of drawing a white ball from the first bag is $\frac{5}{8}$, and from the second, $\frac{3}{5}$. The probability of drawing at least one white ball then reduces to the case of mutually exclusive events after we have properly defined the events. We shall have at least one white ball if we draw (1) a white ball from the first and a black from the second, (2) a red from the first and a white from the second, or (3) a white from both.

The probability of (1) is $\frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4}$. (Independent events)

The probability of (2) is $\frac{3}{8} \cdot \frac{3}{5} = \frac{9}{40}$. (Independent events)

The probability of (3) is $\frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$. (Independent events)

The probability of (1) or (2) or (3) is $\frac{1}{4} + \frac{9}{40} + \frac{3}{8} = \frac{34}{40}$
 $= \frac{17}{20}$. (Mutually exclusive events)

EXERCISE I-3

1. A bag contains 3 white, 9 red, and 12 blue balls. If one ball is drawn, what is the probability that it will be (a) white? (b) Red or blue?

2. If two dice are thrown, what is the probability (a) of throwing a double? (b) Of throwing a 7? (To throw a 7 means that the sum of the two numbers thrown equals 7; e.g., 4 and 3, 3 and 4, 5 and 2, etc.)

3. If a group of 5 is chosen from 4 boys and 8 girls, what is the probability that it will contain (a) exactly 3 boys? (b) Exactly 3 girls?

4. If three books are taken from a group containing 4 Spanish and 6 French books, what is the probability (a) that exactly 2 are French? (b) That at least 2 are French?

5. If 5 coins are thrown, what is the probability that they will fall 2 heads and 3 tails?

6. If 5 coins are thrown, what is the probability that they will fall (a) no tails? (b) Exactly one tail? (c) Exactly two tails? (d) Exactly three tails? (e) Exactly four tails? (f) All tails? Can you devise a check for your answers?

7. Two men go up to their offices in a building having 11 elevators. What is the probability that they use the same elevator?

8. Nine books, two of them algebras, are placed on a shelf by a person who cannot read. What is the probability that the two algebras will be placed side by side?

9. What is the probability of throwing 10 in a single throw with 2 dice?

10. If 2 dice are thrown, what is the probability of throwing (a) 7? (b) 7 or 11? (c) 4 or less?

11. A group of cards consisting of 1 jack, 2 queens, 3 kings, and 4 aces are laid face down. If three are drawn, what is the probability of drawing 1 queen and 2 aces?

12. Using the data in Problem 11, if 3 cards are drawn, what is the probability of drawing at least 2 aces?

13. (a) Smith and Jones match coins for a prize. If the winner matches with Brown, and again the winner matches with Robinson, what is the probability that each will win the prize? (b) If Smith matches with Jones, and Brown matches with Robinson, and the two winners match, what is the probability that each will win the prize?

14. (a) If three men throw coins, what is the probability that one coin falls unlike the other two? (b) If four men throw, what is the probability that one coin falls unlike the other three? (c) If five men throw, what is the probability that one coin falls unlike the other four?

15. Nine names, including Mr. X's, are written on slips of paper and dropped in a hat. Three slips are to be drawn in succession. What is the probability

(a) that Mr. X's name will be drawn?

(b) that his name will be drawn the first time?

(c) that his name will not be drawn the first time, but on the second draw?

(d) that his name will not be drawn the first nor second time, but on the third draw?

16. 49 bottles are standing on a wall; 40 are blue, 9 are red, and 2 of the red bottles have black spots. A marksman fires thrice, and scores three hits. What is the probability that the 2 red bottles with the black spots are still intact?

17. A student takes an examination in a subject about which he knows absolutely nothing. There are 5 questions to be answered "yes" or "no." What is the probability that he will get (a) just 60%? (b) At least 60%?

18. A hand of 5 cards is dealt from a pack of 52. What is the probability that it contains (a) just 3 aces? (b) 5 of the same suit (any suit)? (c) Just 2 sixes and just 2 sevens? (d) 3 of the same kind (i.e., the same rank), and a pair of a different kind? (e) An ace, king, queen, jack, and ten (not necessarily of the same suit)?

19. Three members of the same fraternity are among 20 students who volunteer to usher at a college function. If 8 ushers are chosen, what is the probability that all three are selected?

20. Smith, Jones, Brown, and Robinson sit down to play bridge. What is the probability that Smith and Jones do not sit opposite each other?

21. A bag contains 4 white and 6 red balls, and a second bag contains 10 white and 5 blue balls. If I draw a ball from each bag, what is the probability that I draw (a) a white and a blue? (b) Just one white ball? (c) At least one white ball?

22. The probability that Smith will win a prize is $\frac{1}{5}$; that his wife will win the same prize is $\frac{1}{4}$. What is the probability that one or the other of them will win?

23. The probability that Smith will win a prize is $\frac{1}{5}$; that his wife will win a different prize is $\frac{1}{4}$. What is the probability that at least one of them will win a prize?

24. The probability that a certain athlete will win his letter in football is $\frac{1}{3}$, in basketball $\frac{3}{4}$, and in track $\frac{2}{5}$. What is the probability that he will win just 2 letters during the year?

25. A man has a box with a combination lock controlled by three dials numbered 0, 1, 2, 9. His wife forgets the

combination, but remembers that the first and third numbers are the same, and the middle one is different. What is the probability that she opens the box at the first attempt?

26. A party of 5 politicians are traveling abroad with their wives, and through diplomatic channels they wangle 4 invitations to a tea given by His Royal Highness. If they all draw lots for the invitations, what is the probability that two married couples will attend?

27. 10 persons enter a cafe having 50 vacant chairs. Syrup has been spilled on one of the chairs. If they choose seats at random, what is the probability that someone sits in the syrup?

28. 8 persons, among them 2 brothers, take seats around a circular table. What is the probability that the brothers do not sit side by side?

29. 25 slips numbered from 1 to 25 are dropped into a hat. If 2 are drawn, what is the probability that their sum is odd?

30. 4 men and their wives sit down for 2 tables of bridge. They cut cards for partners, high man playing with high lady, second high man with second high lady, etc. What is the probability that each man and wife start out as partners?

I-9. EMPIRICAL PROBABILITY. Hitherto all of the probabilities that have occurred have been mathematically defined, i.e., an exact number of favorable cases out of an exact number of cases favorable and unfavorable. In everyday life many probabilities are not so precisely defined, but are determined by records of a large number of trials of the same event. If after playing many sets of tennis, A has won $\frac{3}{4}$ of them and B has won $\frac{1}{4}$, we say that if they play again the probability that A will win is $\frac{3}{4}$.

The probability of an event established by a large number of trials is called the empirical (statistical, or experimental) probability of the event. We shall assume that the probabilities thus obtained obey the laws of mathematical probability for mutually exclusive, dependent, and independent events.

I-10. THE AMERICAN EXPERIENCE TABLE OF MORTALITY. The American Experience Table of Mortality is an outstanding example of the practical value of probability as determined by statistics. It is based on the records of American life insurance companies, and is used as a basis for calculating premiums. It is reproduced on page 19. Starting with 100,000 persons alive at the age of 10 years, the Table shows the number of persons that may be expected to live to each age from 10 years to 95.

AMERICAN EXPERIENCE TABLE OF MORTALITY

Age	Number Living	Number Dying	Yearly Probability of Dying	Yearly Probability of Living	Age	Number Living	Number Dying	Yearly Probability of Dying	Yearly Probability of Living
10	100 000	749	0.007 490	0.992 510	55	64 563	1 199	0.018 571	0.981 429
11	99 251	746	0.007 516	0.992 484	56	63 364	1 260	0.019 885	0.980 115
12	98 505	743	0.007 543	0.992 457	57	62 104	1 325	0.021 335	0.978 665
13	97 762	740	0.007 569	0.992 431	58	60 779	1 394	0.022 936	0.977 064
14	97 022	737	0.007 596	0.992 404	59	59 385	1 468	0.024 720	0.975 280
15	96 285	735	0.007 634	0.992 366	60	57 917	1 546	0.026 693	0.973 307
16	95 550	732	0.007 661	0.992 339	61	56 371	1 628	0.028 880	0.971 120
17	94 818	729	0.007 688	0.992 312	62	54 743	1 713	0.031 292	0.968 708
18	94 089	727	0.007 727	0.992 273	63	53 030	1 800	0.033 943	0.966 057
19	93 362	725	0.007 765	0.992 235	64	51 230	1 889	0.036 873	0.963 127
20	92 637	723	0.007 805	0.992 195	65	49 341	1 980	0.040 129	0.959 871
21	91 914	722	0.007 855	0.992 145	66	47 361	2 070	0.043 707	0.956 293
22	91 192	721	0.007 906	0.992 094	67	45 291	2 158	0.047 647	0.952 353
23	90 471	720	0.007 958	0.992 042	68	43 133	2 243	0.052 002	0.947 998
24	89 751	719	0.008 011	0.991 989	69	40 890	2 321	0.056 762	0.943 238
25	89 032	718	0.008 065	0.991 935	70	38 569	2 391	0.061 993	0.938 007
26	88 314	718	0.008 130	0.991 870	71	36 178	2 448	0.067 665	0.932 335
27	87 596	718	0.008 197	0.991 803	72	33 730	2 487	0.073 733	0.926 267
28	86 878	718	0.008 264	0.991 736	73	31 243	2 505	0.080 178	0.919 822
29	86 160	719	0.008 345	0.991 655	74	28 738	2 501	0.087 028	0.912 972
30	85 441	720	0.008 427	0.991 573	75	26 237	2 476	0.094 371	0.905 629
31	84 721	721	0.008 510	0.991 490	76	23 761	2 431	0.102 311	0.897 689
32	84 000	723	0.008 607	0.991 393	77	21 330	2 369	0.111 064	0.888 936
33	83 277	726	0.008 718	0.991 282	78	18 961	2 291	0.120 827	0.879 173
34	82 551	729	0.008 831	0.991 169	79	16 670	2 196	0.131 734	0.868 266
35	81 822	732	0.008 946	0.991 054	80	14 474	2 091	0.144 466	0.855 534
36	81 090	737	0.009 089	0.990 911	81	12 383	1 964	0.158 605	0.841 395
37	80 353	742	0.009 234	0.990 776	82	10 419	1 816	0.174 297	0.825 703
38	79 611	749	0.009 408	0.990 592	83	8 603	1 648	0.191 561	0.808 439
39	78 862	756	0.009 586	0.990 414	84	6 955	1 470	0.211 359	0.788 641
40	78 106	765	0.009 794	0.990 206	85	5 485	1 292	0.235 552	0.764 448
41	77 341	774	0.010 008	0.989 992	86	4 193	1 114	0.265 681	0.734 319
42	76 567	785	0.010 252	0.989 748	87	3 079	933	0.303 020	0.696 980
43	75 782	797	0.010 517	0.989 483	88	2 146	744	0.346 692	0.653 308
44	74 985	812	0.010 829	0.989 171	89	1 402	555	0.395 863	0.604 137
45	74 173	828	0.011 163	0.988 837	90	847	385	0.454 545	0.545 455
46	73 345	848	0.011 562	0.988 438	91	462	246	0.532 466	0.467 534
47	72 497	870	0.012 000	0.988 000	92	216	137	0.634 259	0.365 741
48	71 627	896	0.012 509	0.987 491	93	79	58	0.734 177	0.265 823
49	70 731	927	0.013 106	0.986 894	94	21	18	0.857 143	0.142 857
50	69 804	962	0.013 781	0.986 219	95	3	3	1.000 000	0.000 000
51	68 842	1001	0.014 541	0.985 459					
52	67 841	1044	0.015 389	0.984 611					
53	66 797	1091	0.016 333	0.983 667					
54	65 706	1143	0.017 396	0.982 604					

Example 1. What is the probability that a person entering college at the age of 18, and having no collegiate difficulties, will live to graduate at the age of 22?

Solution. The mortality table shows that of 94,089 persons alive at the age of 18, there will be 91,192 alive at age 22. Hence, the probability that the person in the example will live to graduate is $\frac{91,192}{94,089} = 0.969$.

I-11. REPEATED TRIALS OF AN EVENT. Assuming that the probability of a single occurrence of an event is known, the question arises, what is the probability that it will occur a given number of times in a specified number of trials. Consider first a particular example.

Example 1. What is the probability of throwing 3 twice in 5 throws of a die?

Solution. At any throw the probability of throwing 3 is $\frac{1}{6}$; the probability of not throwing 3 is $\frac{5}{6}$. The probability of throwing 3 twice out of 5 throws in the particular order, success on each of the first two throws and failure on the remaining three, is $\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{5^3}{6^5}$. The number of essentially different orders in which the 3's may be thrown is ${}_5C_2$. Multiplying the number of orders by the probability of any particular order gives the required probability which is

$$p = {}_5C_2 \cdot \frac{5^3}{6^5} = \frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{5 \cdot 5 \cdot 5}{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6} = \frac{625}{3888}.$$

Example 2. What is the probability of throwing 3 at least twice in 5 throws of a die?

Solution. At least twice means two, three, four, or five times. The solution of each case is similar to Example 1, and since they are mutually exclusive, the sum of the several probabilities is the answer. Thus,

$$\begin{aligned} p &= {}_5C_2 \cdot \frac{5^3}{6^5} + {}_5C_3 \cdot \frac{5^2}{6^5} + {}_5C_4 \cdot \frac{5}{6^5} + {}_5C_5 \cdot \frac{1}{6^5} \\ &= \frac{625}{3888} + \frac{125}{3888} + \frac{25}{7776} + \frac{1}{7776} = \frac{763}{3888}. \end{aligned}$$

In general, if p and q denote the probabilities that an event will, and will not, occur, then the probability that the event will occur exactly r times in n trials is

$$P = {}_nC_rp^rq^{n-r}.$$

The factors $p^r q^{n-r}$ represent the probability that the event will occur r times and fail to occur $n - r$ times in a particular order, and the factor ${}_nC_r$ represents the possible number of orders. Compare the structure of this term carefully with the numerical term ${}_5C_2 \cdot \frac{5^3}{6^5}$ in Example 1.

The probability that the event will occur at least r times in n trials is

$$P = {}_nC_rp^r q^{n-r} + {}_nC_{r+1}p^{r+1}q^{n-r-1} + {}_nC_{r+2}p^{r+2}q^{n-r-2} + \dots + {}_nC_np^n.$$

The foregoing results may be summarized and extended by expanding $(q + p)^n$ by the binomial theorem.

$$(q + p)^n = q^n + {}_nC_1pq^{n-1} + {}_nC_2p^2q^{n-2} + \dots + {}_nC_rp^r q^{n-r} + \dots + {}_nC_{n-1}p^{n-1}q + p^n. \quad [I-XIV]$$

The term whose coefficient is ${}_nC_r$ is the probability that the event will occur exactly r times in n trials; the sum of the terms beginning with the ${}_nC_r$ term and continuing through the term p^n is the probability that the event will occur at least r times in n trials; the sum of the terms commencing with q^n and continuing through the ${}_nC_r$ term is the probability that the event will occur at most r times in n trials. Since $q + p = 1$, the left member of Formula [I-XIV] is unity, indicating the certainty that one or another of the possibilities denoted by the terms in the right member must occur.

EXERCISE I-4

In Problems 1 to 8 use the American Experience Table of Mortality.

1. What is the probability (a) that a person alive at age 21 will live to age 70? (b) That a person alive at age 21 will die during the year he is 70?

2. What is the probability (a) that a person alive at age 60 will live to age 70? (b) That he will live to be at least 70, but not 75?

3. A couple marry when each is 25. What is the probability that they will live to celebrate their Golden Wedding Anniversary?

4. Referring to the couple in Problem 3, what is the probability that at least one of them will live to be 75?

5. A man aged 60 has a wife aged 55. He makes out a will leaving his fortune to his wife if she survives him, otherwise it goes to a permanently endowed institution. Assuming

that he dies at 70, what is the probability that the institution will get the fortune?

6. Referring to the couple in Problem 5, what is the probability that the man will die at age 75 and that his wife will get his fortune?

7. A man aged 70 has a wife the same age and a nephew aged 40. His will leaves his property to his wife if she survives him, otherwise to his nephew if he survives. What is the probability that the nephew will inherit during the year he is 45?

8. Three friends aged 20, 21, and 22 all graduate from college the same year. What is the probability that all three will be alive for their 25th reunion?

9. If a coin is tossed 7 times, what is the probability of throwing (a) just 5 tails? (b) At least 5 tails?

10. If dice are tossed, what is the probability that just two 3's will be thrown?

11. When A and B play tennis, the probability that A will win any particular set is $\frac{3}{4}$. If they play 5 sets, what is the probability (a) that A will win just three sets? (b) That B will win just three sets?

12. During a certain season the probability that it will rain in a certain town on any day is $\frac{1}{5}$. If a carnival comes to town for 4 days, what is the probability that they will strike (a) just one rainy day? (b) At most one rainy day?

13. A bag contains 4 red and 2 white balls. If a ball is drawn 5 times, and replaced between each draw, what is the probability of drawing a red ball at least 3 times?

14. If the probability that a certain boat is on time is $\frac{9}{10}$ on any given day, what is the probability that it will be on time just 5 days during a week (7 days)?

15. Five veterans in an Old Soldiers Home have reached the age of 94. Using the Mortality Table, what is the probability that just three will reach the age of 95?

16. At a certain college 60% of each entering class is expected to graduate. If a certain High School sends 6 students, what is the probability that at most 4 will graduate?

Chapter II

DETERMINANTS

II-1. INTRODUCTION. In an elementary way determinants are a short cut for solving systems of simultaneous linear equations; they are also a valuable tool in many advanced branches of mathematics. In this chapter we shall consider the application mentioned first, and develop some of the general properties of determinants to afford the student preparation for their use in topics which he may pursue later.

II-2. DETERMINANTS OF THE SECOND ORDER. Consider the following system of two simultaneous linear equations in x and y . Assume that the equations are independent.

$$\begin{aligned}(1) \quad & a_1x + b_1y = r_1, \\ & a_2x + b_2y = r_2.\end{aligned}$$

Multiplying the first by b_2 and the second by $-b_1$ we have

$$\begin{aligned}a_1b_2x + b_1b_2y &= r_1b_2, \\ -a_2b_1x - b_2b_1y &= -r_2b_1.\end{aligned}$$

Adding, and solving for x ,

$$(2) \quad x = \frac{r_1b_2 - r_2b_1}{a_1b_2 - a_2b_1}.$$

Similarly, multiplying the first of (1) by $-a_2$ and the second by a_1 , and adding to eliminate x , yields

$$(3) \quad y = \frac{a_1r_2 - a_2r_1}{a_1b_2 - a_2b_1}.$$

Examination of (2) and (3) shows that their denominators are identical, and their numerators are both of the same form; i.e., the difference of the products of two pairs of numbers. The notation for such a combination of terms is called a determinant of the second order, and is defined as follows.

Definition. The array of letters $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, preceded and followed by a vertical bar, is a determinant of the second order, and its value is $a_1b_2 - a_2b_1$.

Any one of the letters a_1, b_1, a_2, b_2 , is an element of the determinant; the elements a_1, b_2 constitute the principal diagonal of the determinant; the elements a_1, b_1 and a_2, b_2 form

the first and second rows of the determinant, respectively; and the elements a_1, a_2 and b_1, b_2 form the first and second columns.

Expressed in determinant notation, the solution of (1)

is

$$(4) \quad x = \frac{\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

The determinant in both denominators comprises the coefficients of x and y ; it is called simply the determinant of the coefficients, and is denoted by Δ (Greek letter delta). It is also called the determinant of the system.

Observe also that the numerator of x is similar to Δ , except that the first column a_1, a_2 (the coefficients of x) is replaced by r_1, r_2 (the column of constant terms). Let Δ_1 denote the determinant in which the first column of Δ is replaced by the column of constant terms.

In like manner the numerator of y is formed by replacing the second column of Δ (the coefficients of y) by the column of constant terms. Let Δ_2 denote the determinant in which the second column of Δ is replaced by the column of constant terms.

With the aid of this latter notation we may summarize the solution of (1) in the following

Rule. The solution of $\begin{cases} a_1x + b_1y = r_1 \\ a_2x + b_2y = r_2 \end{cases}$ is given by

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad \text{provided that } \Delta \neq 0.$$

Example 1. Evaluate $\begin{vmatrix} 6 & 12 \\ -3 & -5 \end{vmatrix}$.

Solution. $\begin{vmatrix} 6 & 12 \\ -3 & -5 \end{vmatrix} = (6)(-5) - (-3)(12) = -30 + 36 = 6.$

Example 2. Solve $\begin{cases} 2x + 3y = 0, \\ 5x - 2y = 19. \end{cases}$

$$\text{Solution. } x = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 0 & 3 \\ 19 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}} = \frac{0 - 57}{-4 - 15} = \frac{-57}{-19} = 3.$$

$$y = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 2 & 0 \\ 5 & 19 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}} = \frac{38 - 0}{-19} = \frac{38}{-19} = -2.$$

EXERCISE II-1

Evaluate 1-9.

$$1. \begin{vmatrix} 3 & -4 \\ 1 & 7 \end{vmatrix}.$$

$$2. \begin{vmatrix} 5 & -2 \\ 3 & -4 \end{vmatrix}.$$

$$3. \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}.$$

4. $\begin{vmatrix} 1 & -6 \\ -2 & 1 \end{vmatrix}.$

5. $\begin{vmatrix} 4 & 7 \\ 6 & 8 \end{vmatrix}.$

6. $\begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix}.$

7. $\begin{vmatrix} 0 & 3 \\ 2 & -5 \end{vmatrix}.$

8. $\begin{vmatrix} 8 & -4 \\ 0 & 1 \end{vmatrix}.$

9. $\begin{vmatrix} 0 & 0 \\ -3 & 8 \end{vmatrix}.$

Solve equations 10-12 for k.

10. $\begin{vmatrix} k & 3 \\ 2 & 1 \end{vmatrix} = 0.$

11. $\begin{vmatrix} 10 & 5k \\ 3 & -6 \end{vmatrix} = -45.$

12. $\begin{vmatrix} 1 & 8 \\ 2 & k^2 \end{vmatrix} = 0.$

Solve the following systems of equations by determinants.

13. $\begin{cases} 2x + 5y = 13, \\ x - 3y = 1. \end{cases}$

14. $\begin{cases} 2x + 3y = 11, \\ 4x - 2y = -18. \end{cases}$

15. $\begin{cases} 9x - 8y = 7, \\ 3x + 6y = -2. \end{cases}$

16. $\begin{cases} 8x + 12y = -7, \\ -2x + 4y = 0. \end{cases}$

17. $\begin{cases} 6x + 4y = 15, \\ 3x + 2y = 8. \end{cases}$

18. $\begin{cases} x - 4y = -1, \\ 3x - 20y = 1. \end{cases}$

19. $\begin{cases} 5x - 7y = 9, \\ x + 14y = 4. \end{cases}$

20. $\begin{cases} 16x - 4y = 5, \\ 12x - 3y = 26. \end{cases}$

II-3. DETERMINANTS OF THE THIRD ORDER. A determinant of the third order is defined as follows.

Definition. An array of letters of the form $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$ preceded and followed by a vertical bar, is a determinant of the third order, and its value is

$$(1) \quad a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

The nomenclature of the third order determinant is analogous to that of the second order. a_1, b_2, c_3 constitute the principal diagonal; a_1, b_1, c_1 (etc.) are the first (etc.) row; a_1, a_2, a_3 (etc.) are the first (etc.) column.

Notice the symmetry of terms in (1); each term contains the letters a, b, and c in alphabetical order with the subscripts 1, 2, and 3 occurring in their six possible arrangements. The six terms of (1) may also be described by saying that they constitute all the possible combinations of the nine elements taking one element from each row and each column in every term. Finally, notice that half the terms are positive, half are negative. These observations will be of value when we consider determinants of higher order than the third.

There are various devices which make it unnecessary to memorize (1). One of them is the following.

Rewrite $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The products of the elements along the arrows sloping downward are the three positive terms of (1); the products along the arrows sloping upward are the three negative terms of (1).

Example 1. Evaluate $\begin{vmatrix} 2 & 4 & -1 \\ 6 & -3 & 0 \\ 1 & 2 & -1 \end{vmatrix}$.

Solution. Write $\begin{vmatrix} 2 & 4 & -1 \\ 6 & -3 & 0 \\ 1 & 2 & -1 \end{vmatrix} \begin{vmatrix} 2 & 4 \\ 6 & -3 \\ 1 & 2 \end{vmatrix}$

$$(2)(-3)(-1) + (4)(0)(1) + (-1)(6)(2) - (1)(-3)(-1) - (2)(0)(2) - (-1)(6)(4) = 6 + 0 - 12 - 3 - 0 + 24 = 15.$$

II-4. SOLUTION OF THREE SIMULTANEOUS LINEAR EQUATIONS.

Consider the following system of three simultaneous linear equations which for the present we shall assume are independent.

$$\begin{aligned} (1) \quad & a_1x + b_1y + c_1z = r_1, \\ & a_2x + b_2y + c_2z = r_2, \\ & a_3x + b_3y + c_3z = r_3. \end{aligned}$$

By the process of elimination similar to that employed in the solution of the system (1) in §II-2, it may be shown (see Problem 10 in Exercise II-2) that

$$(2) \quad x = \frac{r_1b_2c_3 + r_3b_1c_2 + r_2b_3c_1 - r_3b_2c_1 - r_1b_3c_2 - r_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3};$$

or,

$$(3) \quad x = \frac{\begin{vmatrix} r_1 & b_1 & c_1 \\ r_2 & b_2 & c_2 \\ r_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

provided the determinant in the denominator is not zero.

If Δ denotes the determinant of the coefficients of x , y , and z in (1), and if Δ_1 denotes the determinant obtained by substituting the column of constant terms for the elements in the first column of Δ , then

$$x = \frac{\Delta_1}{\Delta}, \text{ provided } \Delta \neq 0.$$

Similarly, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$, provided $\Delta \neq 0$, where Δ_2 and Δ_3 denote the determinants obtained by substituting the column of constant terms for the second and third columns of Δ , respectively.

Example 1. Solve for y only: $3x - 2y + z = 10$,
 $2x + y - 4z = -12$,
 $x + 5y - 2z = -15$.

Solution. We remark that the solution of equations by determinants is especially suited for finding any one, or more, of the unknowns when it is unnecessary to find all. Employing the notation above,

$$y = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 10 & 1 \\ 2 & -12 & -4 \\ 1 & -15 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -4 \\ 1 & 5 & -2 \end{vmatrix}} = \frac{72 - 40 - 30 + 12 - 180 + 40}{-6 + 8 + 10 - 1 + 60 - 8} = \frac{-126}{63} = -2.$$

Note. If the unknowns in any problem happen not to be arranged in the same sequence in all the equations of the system, they must first be rearranged to put them in the same order.

EXERCISE II-2

Evaluate the following determinants.

1. $\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 1 & -2 & 5 \\ 4 & 0 & 3 \\ -1 & 6 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 0 & 4 \\ 0 & 3 & -2 \\ 5 & -1 & 0 \end{vmatrix}$

4. $\begin{vmatrix} -1 & 4 & 1 \\ -1 & 3 & 2 \\ 1 & 0 & -2 \end{vmatrix}$

5. $\begin{vmatrix} 0 & -1 & 2 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solve the following equations for k .

7. $\begin{vmatrix} 1 & 0 & -2 \\ 2 & k & -3 \\ 1 & -1 & 3 \end{vmatrix} = 8$

8. $\begin{vmatrix} k & 3 & 0 \\ 1 & 2 & -1 \\ 0 & -3 & 4 \end{vmatrix} = 0$

9. $\begin{vmatrix} k & 2 & 1 \\ -1 & 1 & 3 \\ 0 & -2 & k \end{vmatrix} = 22$

10. Carry out the details of the solution of equations (1) and obtain (2) in §II-4.

11. (a) Write out completely the six terms of Δ_2 in the numerator of y . (b) The same for Δ_3 .

12. Find x and z for the equations in Example 1 in §II-4.

Solve the following equations for x , y , and z by determinants.

$$\begin{aligned} 13. \quad & x + 3y - 2z = 7, \\ & 3x - y + 4z = 1, \\ & 2x + y + 5z = 0. \end{aligned}$$

$$\begin{aligned} 14. \quad & 3x + 2y - 2z = -3, \\ & 5x - z = 0, \\ & x + y + z = 8. \end{aligned}$$

$$\begin{aligned} 15. \quad & 2x - 5y + z - 13 = 0, \\ & x - 4z + y + 6 = 0, \\ & z - x - 2y - 1 = 0. \end{aligned}$$

$$\begin{aligned} 16. \quad & x + z = 0, \\ & 2x - y = 0, \\ & y - 4z = 6. \end{aligned}$$

$$\begin{aligned} 17. \quad & 2x + 3y + 4z = 3, \\ & 4x - 9y + 8z = 1, \\ & x - 2z = 0. \end{aligned}$$

$$\begin{aligned} 18. \quad & x + y + 3z = 0, \\ & x - 2y - 6z = 6, \\ & 2x - 3y - 9z = 10. \end{aligned}$$

$$\begin{aligned} 19. \quad & 4x - y + 6z = 6, \\ & 3x - 2y + z = 3, \\ & x + y + 5z = 0. \end{aligned}$$

$$\begin{aligned} 20. \quad & 3x + 2y + 5z = 1, \\ & x - 3y - 2z = 1, \\ & 5x - y + 4z = 1. \end{aligned}$$

II-5. INVERSIONS. Among the numbers 1, 2, 3, n the precedence of any number before any other, contrary to the natural order, constitutes an inversion. The total number of inversions of an arrangement of numbers is found by counting all of the individual inversions. Thus, 1 3 2 4 5 contains only one inversion (3 precedes 2); 2 1 5 4 3 contains four inversions (2 precedes 1, 5 precedes both 4 and 3, and 4 precedes 3). Similarly, inversions among letters are said to occur when a letter precedes another contrary to alphabetical order.

Certain theorems concerning inversions are desirable before defining the expansion of a determinant of the n th order, and establishing its properties.

Theorem 1. Interchanging two adjacent elements (numbers or letters) in an arrangement increases by one, or decreases by one, the number of inversions in the arrangement.

Proof. Let j and k denote adjacent numbers in the arrangement $AjkZ$ where A denotes the group of numbers preceding j , and Z denotes the group of numbers following k . By interchanging j and k a new arrangement $AkjZ$ is formed. Observe first that the interchange leaves unchanged the number of inversions of both j and k with respect to the numbers in both A and Z ; second, if j and k are in the natural order as given, the interchange increases the number of inversions by one; if j, k as

given constitutes an inversion, interchanging them decreases the total number of inversions by one. A similar argument applies if the elements j and k denote letters.

Theorem 2. Interchanging any two elements in an arrangement of elements (numbers or letters) changes the number of inversions by an odd number.

Proof. Let j and k be any numbers in $AjMkZ$, an arrangement of numbers where A and Z have the same meaning as in Theorem 1, and M denotes the elements, say m in number, lying between j and k . To accomplish the interchange of j and k , let k be interchanged one step at a time with the m elements of M . Then one more interchange, making a total of $m + 1$, will put k in the position of j which, in turn, by m further steps may be carried back to the original position of k . The number of inversions is thus increased or decreased by unity a total of $2m + 1$ times, and therefore differs from its original value by an odd number not exceeding $2m + 1$.

Before stating the next theorem consider a product of elements consisting of a letter with a subscript; e.g., $a_1c_3b_2d_4$. It is understood that such a product contains the first n letters of the alphabet without repetition, together with the numbers 1 to n , inclusive, as subscripts. In general such a product will possess a certain number of inversions with respect to the letters and a different number of inversions with respect to the subscripts. The following theorems refer to products of this type.

Theorem 3. Interchanging two adjacent factors of a product changes the sum of the inversions of letters and subscripts combined by +2, 0, or -2.

Proof. From Theorem 1 the number of inversions of the letters among themselves is increased by +1; similarly the number of inversions of the subscripts is increased by +1. Hence, the sum of the inversions of the letters and subscripts combined is changed by +1 +1 which equals +2, 0, or -2.

Theorem 4. Interchanging any two factors of a product changes the sum of the inversions of letters and subscripts combined by an even number or zero.

Proof. This follows from Theorem 2 in a manner analogous to the way Theorem 3 followed from Theorem 1. By Theorem 2 the number of inversions of letters and subscripts separately are changed by an odd number; hence, their sum combined is changed by the sum of two odd numbers which is an even number or zero.

II-6. THE DETERMINANT OF THE n TH ORDER. We can now define a determinant of the n th order together with its expansion.

Definition. An array of n^2 elements, preceded and followed by a vertical bar, of the form

$$\begin{vmatrix} a_1 & b_1 & \dots & n_1 \\ a_2 & b_2 & \dots & n_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & \dots & n_n \end{vmatrix}$$

is called a determinant of the n th order. Its expansion is defined to be the algebraic sum of all the products which may be formed

A. By taking as factors one and only one element from each row and column, and

B. By giving to each such product a plus or a minus sign according as the number of inversions of the symbols which designate the rows (here the subscripts) is even or odd when the symbols for the columns (here the letters) are written in the same order as they appear in the first row.

The terms principal diagonal, rows, and columns have meanings similar to those for determinants of the third order.

The student should verify immediately that the definition holds for determinants of the second and third orders. For convenience their values are restated here.

$$(1) \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

$$(2) \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3.$$

The expansion of a determinant in accordance with the above definition produces $n!$ terms. To see this, choose any of the elements in the first column as a factor of a term. There are n possible choices. The choice eliminates the row to which the chosen element belongs, but leaves $n - 1$ eligible elements to choose from the second column. When this choice has been made there are $n - 2$ eligible elements in the third column, and so on. Thus,

$$n(n - 1)(n - 2) \dots 3 \cdot 2 \cdot 1 = n!$$

different products may be formed.

Since the evaluation of a determinant of the 4th order, in general, contains $4! = 24$ terms; of the 5th order, $5! = 120$ terms; etc., it should not be surprising to learn that there is

no scheme of diagonals for their evaluation. However, by studying some general properties of determinants the labor of evaluating them is brought within moderate limits.

II-7. PROPERTIES OF DETERMINANTS.

Property 1. If the rows and columns of a determinant are interchanged (first row with first column, second row with second column, etc.), the value of the determinant is unchanged.

Illustration.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

Proof. We shall give the argument for a determinant of the fourth order. Let D denote the left member of the illustration and D' the right.

Consider a typical product, say $a_1b_3c_4d_2$, which, except possibly for sign, will be a term of the expansions of both D and D' . The numerical value of the product is the same in both cases, and it remains to consider its sign. By repeated applications of Theorem 4 in §II-5, $a_1b_3c_4d_2$ may be rearranged so that its subscripts are in the natural order; i.e., in the order $a_1d_2b_3c_4$, and the sum of the inversions of the letters and subscripts combined is left even or odd by the rearrangement according as their sum initially was even or odd. But there are no inversions in the letters of $a_1b_3c_4d_2$, nor in the subscripts of $a_1d_2b_3c_4$, so it follows that if the number of inversions in the subscripts of $a_1b_3c_4d_2$ is even (or odd), the number of inversions of the letters of $a_1d_2b_3c_4$ must be even (or odd). Since the subscripts are the row-symbols of $a_1b_3c_4d_2$ in D , and the letters are the row-symbols of $a_1d_2b_3c_4$ in D' , it follows from the definition of the expansion of a determinant that the product has the same sign for both expansions.

A similar argument holds for any term of a determinant of any order.

Corollary. Properties of determinants which hold for rows, hold also for columns, and vice versa.

This property saves unnecessary duplication in proving properties for both rows and columns.

Property 2. Interchanging two rows (columns) of a determinant changes the sign of the determinant.

Illustration:

$$\begin{vmatrix}
 a_1 & b_1 & c_1 & d_1 \\
 \dots & \dots & \dots & \dots \\
 a_j & b_j & c_j & d_j \\
 \dots & \dots & \dots & \dots \\
 a_k & b_k & c_k & d_k \\
 \dots & \dots & \dots & \dots \\
 a_n & b_n & c_n & d_n
 \end{vmatrix}
 = -
 \begin{vmatrix}
 a_1 & b_1 & c_1 & d_1 \\
 \dots & \dots & \dots & \dots \\
 a_k & b_k & c_k & d_k \\
 \dots & \dots & \dots & \dots \\
 a_j & b_j & c_j & d_j \\
 \dots & \dots & \dots & \dots \\
 a_n & b_n & c_n & d_n
 \end{vmatrix}
 .$$

Proof. Using the determinants in the illustration, interchanging the j th and k th rows does not affect the order of the column-symbols in the first row. Hence, any product in the expansion of the left member becomes the corresponding product in the expansion of the right member, except possibly for sign, by merely interchanging its subscripts j and k . By Theorem 2 in §II-5 such an interchange changes the number of inversions of the subscripts by an odd number, and by the definition of the expansion of a determinant changes the sign of the product. Since this is true for every product in the left member, its expansion is the negative of the expansion of the determinant in the right member.

Property 3. If two rows (columns) of a determinant are identical, the value of the determinant is zero.

Proof. Let Δ denote the value of the determinant. If the identical rows are interchanged, the value of the resulting determinant is $-\Delta$. But if the interchanged rows are identical, the arrangement of the determinant is unaltered. Hence, $-\Delta = \Delta$, and $2\Delta = 0$; that is, $\Delta = 0$.

Property 4. If all the elements in any column (row) of a determinant are multiplied by m , the value of the determinant is multiplied by m .

Illustration.

$$\begin{vmatrix}
 ma_1 & b_1 & c_1 \\
 ma_2 & b_2 & c_2 \\
 ma_3 & b_3 & c_3
 \end{vmatrix}
 = m
 \begin{vmatrix}
 a_1 & b_1 & c_1 \\
 a_2 & b_2 & c_2 \\
 a_3 & b_3 & c_3
 \end{vmatrix}
 .$$

Proof. This property follows from the definition of the expansion of a determinant. Each of the $n!$ terms of the expansion of the determinant will contain m as a factor; hence, the value of the original determinant is multiplied by m .

Corollary. If all the elements of any column (row) of a determinant contain a common factor, this factor may be removed by writing it as a coefficient of the determinant.

Corollary. If all the elements of any column (row) of a determinant are zero, the value of the determinant is zero.

Property 5. If all the elements of any column (row), say the i th, of a determinant consist of the sum of two quantities, the determinant may be written as the sum of two determinants whose i th columns are all the first members and all the second members of the sums, respectively.

$$\text{Illustration. } \begin{vmatrix} a_1' + a_1'' & b_1 & c_1 \\ a_2' + a_2'' & b_2 & c_2 \\ a_3' + a_3'' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 \\ a_2'' & b_2 & c_2 \\ a_3'' & b_3 & c_3 \end{vmatrix}.$$

Proof. This follows at once from the definition of the expansion of a determinant. Each element a_i in the $n!$ terms of the normal expansion is replaced by the binomial $(a_i' + a_i'')$. Multiplying out each binomial yields $2(n!)$ terms which are equal to the sum of the values of the determinants as stated.

Property 6. If in any determinant the elements of any column (row) are all multiplied by the same constant and added to the corresponding elements of any other column (row), the value of the determinant is unchanged.

The following illustration shows the meaning of Property 6, and indicates the method of proof.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kc_1 & b_1 & c_1 \\ a_2 + kc_2 & b_2 & c_2 \\ a_3 + kc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} = \Delta + 0.$$

Proof. The proof of Property 6 depends on the successive applications of Properties 5, 4, and 3. By Property 5 the modified determinant may be expressed as the sum of two determinants, one of which is the original determinant, and the other is a constant times a determinant (Property 4) having two identical columns. By Property 3 the value of the latter is zero.

II-8. DEVELOPMENT OF A DETERMINANT BY ITS MINORS. We shall illustrate this operation using a determinant of the third order, and then consider the general case. Reviewing the familiar

$$(1) \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3,$$

note that the expansion may be regrouped so that

$$(2) \Delta = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1); \text{ or}$$

$$(3) \quad \Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

It is easily seen that the determinant $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is obtained from Δ by striking out the row and column in which a_1 appears. It is called the minor of a_1 ; similarly, the other determinants in (3) are the minors of a_2 and a_3 , respectively. The minors of a_1, a_2, a_3 are denoted by A_1, A_2, A_3 , respectively, and (3) may be written as

$$(4) \quad \Delta = a_1A_1 - a_2A_2 + a_3A_3.$$

In (4) the determinant Δ is said to be developed according to the minors of the first column. By factoring the right member of (1) in different orders it may be verified that Δ may be developed in a total of six ways; i.e., by the minors of any of the three rows or three columns. Samples are

$$(5) \quad (\text{By the second row}) \quad \Delta = -a_2A_2 + b_2B_2 - c_2C_2.$$

$$(6) \quad (\text{By the third column}) \quad \Delta = c_1C_1 - c_2C_2 + c_3C_3.$$

The student will have noticed already in the several developments listed that the algebraic signs of the elements are sometimes changed and sometimes unchanged. We state here the rule leaving its proof to be considered in the general case.

Rule. A determinant is equal to the sum of the products of each element in any column (row) multiplied by its minor, the sign of the element being left unchanged when the sum of its row and column is even, and changed when this sum is odd.

Example. c_2 lies in the 2nd row and 3rd column. The sum of $2 + 3 = 5$, an odd number. Hence c_2 appears with its sign changed as illustrated in both (5) and (6) above.

The general case. Let Δ denote a determinant of the n th order.

The minor of any element of a determinant is the determinant that results by striking out the row and column in which the element lies.

$$\Delta = \begin{vmatrix} a_1 & b_1 & \dots & n_1 \\ a_2 & b_2 & \dots & n_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & \dots & n_n \end{vmatrix}$$

We shall denote the minor of any element by using the corresponding capital letter with the same subscript.

Having in mind the expansion of Δ as explained in §II-6, we see that a_1 appears as a factor in $(n-1)!$ terms, and that these $(n-1)!$ terms constitute the minor of a_1 ; i.e., they contain the letters $b \dots n$ in alphabetical order with all possible arrangements of the subscripts. Furthermore, a_1 occupies the first position in each sequence in which it occurs since a is the first letter of the alphabet. Therefore, the removal of a_1 as a factor will not effect the inversions among the subscripts of terms of the expansion containing a_1 . Thus the product $a_1 A_1$ accounts for all the terms of Δ containing a_1 .

The treatment of any other element may be made to depend on the treatment of a_1 if we first carry out the interchanges of rows and columns necessary to bring the element into the position occupied by a_1 . Let k_j be the element in the k th column and j th row. Interchanging the j th row with the one above it $j-1$ times in succession will bring it to the top row, and interchanging columns $k-1$ times to the left in succession will bring the element to the a_1 position. The total number of interchanges is $j+k-2$. If $j+k-2$, or, what is the same thing, if simply $j+k$ is even, the expansion remains unchanged, and the product $k_j K_j$ denotes the terms of Δ in which the element k_j appears; if $j+k$ is odd, the expansion is changed in sign, and $-k_j K_j$ denotes the terms of Δ in which the element k_j appears.

Corollary. If in the development of a determinant by the minors of any column (row), the elements of the column (row) are replaced by the elements of another column (row) of the determinant, the resulting development vanishes.

Illustration. Using (4) above, $\Delta = a_1 A_1 - a_2 A_2 + a_3 A_3$. The corollary states that if a_1, a_2, a_3 are replaced by, say b_1, b_2, b_3 , the result vanishes. Let

$$(7) \quad \Delta' = b_1 A_1 - b_2 A_2 + b_3 A_3.$$

$$\text{Then } \Delta' = \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0, \text{ having two columns identical.}$$

The general case behaves in a similar way.

As an aid to the memory in keeping track of the sum of rows and columns, the diagram at the right is useful; an element is left unchanged in sign if it appears where a sign of the diagram is +; but the element is changed in sign if it appears where a sign is -.

$$\begin{vmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

II-9. EVALUATION OF A DETERMINANT. We shall next consider a way for evaluating determinants of order higher than the third containing numerical elements. In general it is done by Property 6 and developing by minors. Obviously it is desirable to have as many zero elements as possible in some row or column. These are obtained by Property 6. The development by minors follows.

Example. Evaluate:

$$\begin{vmatrix} 2 & 3 & -1 & 2 \\ -2 & 4 & 0 & 5 \\ -6 & 2 & 2 & 8 \\ 8 & 10 & -3 & 3 \end{vmatrix}.$$

Solution. Notice first that the elements in the 3rd row contain 2 as a factor, and that the 3rd column seems most promising for development by minors because it contains a zero already, and a -1. (Next to zero, 1 and -1 are likely to be most useful elements.) While not necessary, it is a good idea to make a record of operations as they are performed to help in the clerical labor. The record below starts by removing the factor 2 from the third row of the given determinant.

$$2 \begin{vmatrix} 2 & 3 & -1 & 2 \\ -2 & 4 & 0 & 5 \\ -3 & 1 & 1 & 4 \\ 8 & 10 & -3 & 3 \end{vmatrix} \begin{array}{l} \leftarrow \text{unchanged} \rightarrow \\ \leftarrow \text{unchanged} \rightarrow \\ \leftarrow \text{add row 1} \rightarrow \\ \leftarrow \text{add } -3 \text{ times row 1} \rightarrow \end{array} = 2 \begin{vmatrix} -2 & 3 & -1 & 2 \\ -2 & 4 & 0 & 5 \\ -1 & 4 & 0 & 6 \\ 2 & 1 & 0 & -3 \end{vmatrix}$$

(sign unchanged)

$$= 2 \left[(-1) \begin{vmatrix} -2 & 4 & 5 \\ -1 & 4 & 6 \\ 2 & 1 & -3 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & 6 \\ 2 & 1 & -3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 3 & 2 \\ -2 & 4 & 5 \\ 2 & 1 & -3 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 2 \\ -2 & 4 & 5 \\ -1 & 4 & 6 \end{vmatrix} \right]$$

$$= -2 \begin{vmatrix} -2 & 4 & 5 \\ -1 & 4 & 6 \\ 2 & 1 & -3 \end{vmatrix} - 2 \begin{vmatrix} -2 & 4 \\ -1 & 4 \\ 2 & 1 \end{vmatrix} = -54.$$

II-10. SOLUTION OF A SYSTEM OF n SIMULTANEOUS LINEAR EQUATIONS. Let (1) be a system of n simultaneous linear equations in n unknowns x, y, z, \dots

$$\begin{aligned}
 (1) \quad & a_1x + b_1y + c_1z + \dots = r_1, \\
 & a_2x + b_2y + c_2z + \dots = r_2, \\
 & \dots = \dots \\
 & \dots = \dots \\
 & a_nx + b_ny + c_nz + \dots = r_n.
 \end{aligned}$$

Analogous to the solutions for systems of two and three equations, let Δ be the determinant of the coefficients of the unknowns, and Δ_i the determinant obtained by substituting the column of constants for the column of coefficients of the i th unknown. The values of the unknowns are given by

$$(2) \quad x = \frac{\Delta_1}{\Delta}; \quad y = \frac{\Delta_2}{\Delta}; \quad i\text{th unknown} = \frac{\Delta_i}{\Delta}; \quad \text{provided } \Delta \neq 0.$$

To obtain the results shown in (2), multiply the first equation of (1) by A_1 (the minor of a_1 in Δ), the second equation by $-A_2$, the third by A_3 , and so on. This gives

$$\begin{aligned}
 (3) \quad & a_1A_1x + b_1A_1y + c_1A_1z \dots = r_1A_1, \\
 & -a_2A_2x - b_2A_2y - c_2A_2z \dots = -r_2A_2, \\
 & \dots = \dots \\
 & \dots = \dots \\
 & +a_nA_nx + b_nA_ny + c_nA_nz \dots = +r_nA_n.
 \end{aligned}$$

Adding,

$$(a_1A_1 - a_2A_2 + \dots + a_nA_n)x = r_1A_1 - r_2A_2 + \dots + r_nA_n,$$

$$\text{or,} \quad \Delta \cdot x = \Delta_1.$$

$$\text{Then, if } \Delta \neq 0, \quad x = \frac{\Delta_1}{\Delta}.$$

In the addition which follows (3) the coefficients of y , z , and of the other unknowns are zero being of the form discussed in the corollary in §II-8.

Similar procedure with the minors of b_1 , b_2 , b_3 , and so on, yields similar results for y and the other unknowns.

The foregoing demonstration shows the form of the solutions of the unknown if they exist; to complete the demonstration it would be necessary to substitute their values in (1) and show that the equations are satisfied. This we shall omit; the complete details may be found in books devoted to Theory of Equations.

Example. Solve for z by determinants, given the equations

$$\begin{aligned}
 x + y + z + u &= 5, \\
 2x - y - z + u &= -4, \\
 x - 2y + 2z - u &= 0, \\
 x - y + z - u &= 1.
 \end{aligned}$$

Solution.

$$\begin{aligned}
 z &= \frac{\begin{vmatrix} 1 & 1 & 5 & 1 \\ 2 & -1 & -4 & 1 \\ 1 & -2 & 0 & -1 \\ 1 & -1 & 1 & -1 \end{vmatrix} \begin{array}{l} \leftarrow \text{Add row 4} \rightarrow \\ \leftarrow \text{Add row 4} \rightarrow \\ \leftarrow \text{Sub. row 4} \rightarrow \\ \leftarrow \text{Unchanged} \rightarrow \end{array}}{\begin{vmatrix} 2 & 0 & 6 \\ 3 & -2 & -3 \\ 0 & -1 & -1 \end{vmatrix} \begin{array}{l} \\ (-1) \\ \end{array}} \\
 &= \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -1 & 1 & -1 \end{vmatrix} \begin{array}{l} \leftarrow \text{Add row 4} \rightarrow \\ \leftarrow \text{Add row 4} \rightarrow \\ \leftarrow \text{Sub. row 4} \rightarrow \\ \leftarrow \text{Unchanged} \rightarrow \end{array}}{\begin{vmatrix} 2 & 0 & 2 \\ 3 & -2 & 0 \\ 0 & -1 & 1 \end{vmatrix} \begin{array}{l} \\ (-1) \\ \end{array}}
 \end{aligned}$$

$$z = \frac{(-1)(-20)}{(-1)(-10)} = 2.$$

EXERCISE II-3

Evaluate the following determinants.

$$\begin{aligned}
 1. & \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \quad 2. \begin{vmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{vmatrix} \quad 3. \begin{vmatrix} 1 & 2 & 2 & 1 \\ 3 & 1 & 0 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 4. & \begin{vmatrix} 2 & 3 & -2 & 0 \\ 4 & -2 & 0 & 3 \\ -3 & 0 & 2 & 1 \\ 0 & 2 & -3 & 4 \end{vmatrix} \quad 5. \begin{vmatrix} 25 & 6 & 2 & 6 \\ 15 & 8 & -1 & -3 \\ 20 & 4 & 3 & -2 \\ 10 & 10 & 0 & 3 \end{vmatrix} \quad 6. \begin{vmatrix} 8 & 3 & -3 & 2 \\ 4 & -2 & 0 & -3 \\ 16 & 5 & -2 & -12 \\ 12 & 10 & 8 & 6 \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 7. & \begin{vmatrix} 2 & 1 & -3 & -2 \\ -4 & -2 & 7 & 0 \\ 3 & 1 & -2 & -2 \\ 1 & 2 & 3 & 3 \end{vmatrix} \quad 8. \begin{vmatrix} 6 & 5 & 4 & 3 \\ 5 & 4 & 3 & 2 \\ 2 & 3 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{vmatrix} \quad 9. \begin{vmatrix} 4 & 3 & 3 & -2 \\ -6 & -6 & -5 & 3 \\ 2 & 5 & 4 & -3 \\ 3 & 2 & 2 & 2 \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
 10. & \begin{vmatrix} 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 \end{vmatrix} \quad 11. \begin{vmatrix} 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 2 & 2 \\ 2 & 0 & 2 & 1 & 1 \\ 0 & 2 & 3 & 0 & 1 \end{vmatrix}
 \end{aligned}$$

$$12. \begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 2 & 1 & 2 \end{vmatrix}$$

13. Complete the solution for x , y , and u in the Example of §II-10.

14. Referring to the system of equations (3) in §II-10, write out the details to show (a) $y = \frac{\Delta_2}{\Delta}$; (b) $z = \frac{\Delta_3}{\Delta}$.

Solve the following systems of equations by determinants.

$$15. \begin{aligned} x + y + z + u &= 5, \\ 4x - y - u &= 4, \\ 3x + 3z + u &= 6, \\ x - y + z &= 0. \end{aligned}$$

$$16. \begin{aligned} x + y + z &= 2, \\ x + y + u &= 3, \\ x + z + u &= 6, \\ y + z + u &= 1. \end{aligned}$$

$$17. \begin{aligned} x + 2y + 3z + u &= 2, \\ x - 6z + u &= -2, \\ x - 2y &= 0, \\ x - 4y - 2u &= 1. \end{aligned}$$

$$18. \begin{aligned} x - y + z + 4u &= 3, \\ x - y - z &= 0, \\ 2x + y - 5z &= 3, \\ z - 4u &= 0. \end{aligned}$$

$$19. \begin{aligned} x + z &= 0, \\ x - 2y + u &= 0, \\ y - z - u &= 0, \\ 2x - y + z &= 1. \end{aligned}$$

$$20. \begin{aligned} x + y + z + u &= 3, \\ 2x - 3y - 2z &= -4, \\ x + z - 3u &= 1, \\ y - 2z + u + 2 &= 0. \end{aligned}$$

$$21. \begin{aligned} x + y - u &= 2, \\ 2y + z + w &= 3, \\ x + z - w &= 0, \\ y + z - 2u &= 5, \\ x - u - w &= 0. \end{aligned}$$

$$22. \begin{aligned} x + y - z &= -1, \\ x + u - w &= 0, \\ y - z + u &= 0, \\ x + y - w &= -2, \\ z + u - w &= 2. \end{aligned}$$

II-11. HOMOGENEOUS LINEAR EQUATIONS. A linear equation in any number of unknowns is homogeneous if it has no constant term; a system of linear equations is a homogeneous system if each of its members is homogeneous.

We shall consider in detail the cases which may arise for a system of three homogeneous equations in three unknowns. Let such a system be

$$(1) \quad \begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \\ a_3x + b_3y + c_3z &= 0. \end{aligned}$$

Case 1. It is seen immediately by inspection that

$$(2) \quad x = 0, y = 0, z = 0$$

is a solution of the system. It is called the zero solution, or trivial solution; and, in general, it is the only solution of such a system. However, other solutions are not impossible.

Case 2. If we write the solution of (1) for x using determinants we obtain $x = \frac{\Delta_1}{\Delta}$. Δ_1 , having a column of zeros, is zero; hence, $x = 0$ (Case 1) unless Δ also equals zero, and in this case the solution $\frac{\Delta_1}{\Delta}$ is meaningless. Of course, when $\Delta = 0$, the solutions for y and z , viz., $y = \frac{\Delta_2}{\Delta}$, and $z = \frac{\Delta_3}{\Delta}$, respectively, are also meaningless. In this case where $\Delta = 0$ it may happen that the minor of some element of Δ is not zero. When this occurs it is possible to solve for two of the unknowns in terms of the third; or, what is the same thing, for the ratio of the unknowns.

The process is illustrated by the following example.

$$(3) \quad \begin{array}{rcl} x - 2y + z & = & 0, \\ 2x + 3y - z & = & 0, \\ 4x - y + z & = & 0. \end{array} \quad \Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \\ 4 & -1 & 1 \end{vmatrix} = 0.$$

By inspection the minor $\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$ in the upper left-hand corner of Δ is not zero. Its value is 7. This minor comes from the coefficients of x and y in the first two members of (3). Write these equations by themselves, and transpose their z -terms to the right members; that is,

$$(4) \quad \begin{array}{rcl} x - 2y & = & -z, \\ 2x + 3y & = & z. \end{array}$$

Solving (4) in the usual way, treating the right members as constants, gives x and y in terms of z .

$$(5) \quad x = \frac{\begin{vmatrix} -z & -2 \\ z & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}} = \frac{-3z + 2z}{3 + 4} = \frac{-z}{7}; \quad y = \frac{\begin{vmatrix} 1 & -z \\ 2 & z \end{vmatrix}}{7} = \frac{z + 2z}{7} = \frac{3z}{7}.$$

Selecting any other non-zero minor of the Δ of (3), and applying treatment similar to (4), would yield equivalent results. (See NOTE following equation (4) in the next section.)

The results in (5) may be put in many forms as follows.

$$(6) \left\{ \begin{array}{l} x = -\frac{z}{7}, y = \frac{3z}{7} \text{ may also be written } -7x = z, \frac{7}{3}y = z; \\ \text{or, } -7x = \frac{7}{3}y = z; \text{ or, } -21x = 7y = 3z; \text{ or, dividing by } -21, \\ \frac{x}{1} = \frac{y}{-3} = \frac{z}{-7}; \text{ or, } x:y:z = 1:-3:-7. \end{array} \right.$$

As a check, the results in (5) should be substituted in the remaining member of (3): $4(-\frac{z}{7}) - \frac{3z}{7} + z = \frac{-7}{7}z + z = 0$.
Check.

A system of equations having a solution of the kind just demonstrated for (3) is said to have a non-trivial solution.

Case 3. If for the system (1), $\Delta = 0$, and if the minors of all the elements of Δ are zero, then any of the unknowns may be expressed in terms of the other two. This happens when the coefficients of the three equations are proportional as follows:

$$a_1:a_2:a_3 = b_1:b_2:b_3 = c_1:c_2:c_3.$$

II-12. SYSTEMS OF LINEAR EQUATIONS HAVING DIFFERENT NUMBERS OF UNKNOWN AND EQUATIONS. We shall consider briefly a few cases in which the number of unknowns in a system differs from the number of equations.

Case 1. Suppose we have a system of \underline{m} equations in \underline{n} unknowns, and $\underline{m} > \underline{n}$. If \underline{n} of the \underline{m} equations are consistent, they may be solved; and if, upon substitution, the solution satisfies the remaining $\underline{m} - \underline{n}$ equations, the system is consistent. The case where $\underline{m} = \underline{n} + 1$ is especially interesting.

Case 2. Consider the special case of a system of three equations in two unknowns. Let the system be

$$(1) \quad \begin{aligned} a_1x + b_1y + k_1 &= 0, \\ a_2x + b_2y + k_2 &= 0, \\ a_3x + b_3y + k_3 &= 0. \end{aligned}$$

Assuming that the second and third equations are consistent,

$$(2) \quad x = \frac{\begin{vmatrix} -k_2 & b_2 \\ -k_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_2 & -k_2 \\ a_3 & -k_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

Substituting these values in the first equation, and clearing fractions.

$$(3) \quad a_1 \begin{vmatrix} -k_2 & b_2 \\ -k_3 & b_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & -k_2 \\ a_3 & -k_3 \end{vmatrix} + k_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0,$$

which may be rearranged as

$$a_1 \begin{vmatrix} b_2 & k_2 \\ b_3 & k_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & k_2 \\ a_3 & k_3 \end{vmatrix} + k_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is seen to be the expansion of

$$(4) \quad \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} = 0.$$

Equation (4) was obtained on the assumption that the second and third equations of (1) are consistent. If any other pair is assumed to be consistent, the same result is obtained.

In general, by a similar argument, a system of $n + 1$ equations in n unknowns is consistent if the determinant of the coefficients and constant terms is zero, provided, also, that some set of n of the equations is consistent.

NOTE. If k_1, k_2, k_3 are made the coefficients of the variable z , system (1) in this section is equivalent to system (1) in the preceding section. The argument used to establish (4) may then be used to establish the facts mentioned in solving the example under Case 2 in the preceding section; viz., that the determinant of the coefficients must be zero, and that using any one of the 2-rowed, non-zero minors of this determinant will yield results equivalent to those obtained by using any other.

Case 3. If the number of unknowns exceeds the number of equations, there are, in general, an infinite number of solutions. We illustrate using a system of two equations in three unknowns.

$$(5) \quad \begin{aligned} 2x + 3y - z &= 5, \\ x + 2y - 3z &= 4. \end{aligned}$$

Transpose the z -terms, and treat $z + 5, 3z + 4$ as constant terms.

$$(6) \quad x = \frac{\begin{vmatrix} z + 5 & 3 \\ 3z + 4 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 2 & z + 5 \\ 1 & 3z + 4 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}}.$$

Reducing,

$$(7) \quad x = -7z - 2, \quad y = 5z + 3.$$

Equations (7) are formulas for the solutions of system (5) and by assigning values arbitrarily to z , an infinite number of solutions may be obtained; e.g., $z = -2, x = 12, y = -7$ is a particular solution.

Case 4. We close the chapter by considering the case of two homogeneous equations in three unknowns. Let the system be

$$(8) \quad a_1x + b_1y + c_1z = 0,$$

$$a_2x + b_2y + c_2z = 0.$$

Transposing the z -terms, and solving as in Case 3,

$$(9) \quad x = \frac{\begin{vmatrix} -c_1z & b_1 \\ -c_2z & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} z; \quad y = \frac{\begin{vmatrix} a_1 & -c_1z \\ a_2 & -c_2z \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} z.$$

From (9) may be obtained the continued proportion

$$(10) \quad x:y:z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Equation (10) is particularly easy to remember if it be noted that in the right member the second column of one determinant becomes the first column of the next in a cyclical arrangement. This formula may be used to advantage as an alternative to steps (4), (5), and (6) in §II-11.

EXERCISE II-4

Solve the following systems of homogeneous equations, finding non-trivial solutions where possible.

1. $x + 2y - 7z = 0,$
 $2x - y - 4z = 0,$
 $3x - 5y + z = 0.$
2. $2x - 3y - 4z = 0,$
 $5x + y + 7z = 0,$
 $x + z = 0.$
3. $3x - y + 2z = 0,$
 $x + 5y - z = 0,$
 $4x + 4y + 3z = 0.$
4. $x + 2y = 0,$
 $3x - 4y + 5z = 0,$
 $2x + 2y + z = 0.$
5. $x + y - z = 0,$
 $2x + 4y + 3z = 0,$
 $4x + 6y + z = 0.$
6. $x + y - 5z = 0,$
 $3x - y + 2z = 0,$
 $4x + 3y - 3z = 0.$
7. $2x - 5y + z = 0,$
 $x + 3y + 17z = 0,$
 $x - 2y + 2z = 0.$
8. $4x - 3z = 0,$
 $2x + y - z = 0,$
 $2y + z = 0.$
9. $4x + y = 0,$
 $y + 2z = 0,$
 $2x - z = 0.$
10. $2x - y + 5z = 0,$
 $x - 3y + 2z = 0,$
 $3x + y + 10z = 0.$
11. $2x - 3y + 5z = 0,$
 $4x - 6y - z = 0,$
 $8x - 12y + 4z = 0.$
12. $x - 2y + 4z = 0,$
 $3x - y + 2z = 0,$
 $4x - 3y + 6z = 0.$

Find a solution of the following equations if possible.

$$\begin{aligned}13. \quad 2x + 3y &= 3, \\ 5x - 2y &= 17, \\ x + 4y &= -1, \\ 3x - y &= 10.\end{aligned}$$

$$\begin{aligned}14. \quad 3x + y &= 9, \\ x - 2y &= -4, \\ 5x - 3y &= 1, \\ 2x + 3y &= 0.\end{aligned}$$

Test the following systems for consistency, and if consistent, find solutions.

$$\begin{aligned}15. \quad 2x - y + 1 &= 0, \\ 3x - 2y + 4 &= 0, \\ 3x - y - 1 &= 0.\end{aligned}$$

$$\begin{aligned}16. \quad x + 2y + 2 &= 0, \\ 3x - y - 15 &= 0, \\ 5x + 3y - 11 &= 0.\end{aligned}$$

$$\begin{aligned}17. \quad 3x + y - 1 &= 0, \\ 8x + 3y - 2 &= 0, \\ x + 2y + 10 &= 0.\end{aligned}$$

$$\begin{aligned}18. \quad 4x + y - 1 &= 0, \\ x - 2y + 11 &= 0, \\ x + 7y - 34 &= 0.\end{aligned}$$

Find k , given that each of the following systems is consistent.

$$\begin{aligned}19. \quad x + ky + 1 &= 0, \\ 2x + 3y - 11 &= 0, \\ 3x - 2y - 10 &= 0.\end{aligned}$$

$$\begin{aligned}20. \quad 5x + y - 2 &= 0, \\ 8x + 3y + k &= 0, \\ x + 2y + 5 &= 0.\end{aligned}$$

$$\begin{aligned}21. \quad 2x - 3y - k &= 0, \\ 3x + 2y - 4 &= 0, \\ kx - 4y - 18 &= 0.\end{aligned}$$

$$\begin{aligned}22. \quad 3x - 2y + k &= 0, \\ x + 3y - 7 &= 0, \\ 5x + 4y - 13 &= 0.\end{aligned}$$

Find formulas for the solutions of the following systems, and several particular solutions of each.

$$\begin{aligned}23. \quad x + y - z &= 4, \\ x - y - 3z &= 2.\end{aligned}$$

$$\begin{aligned}24. \quad 3x - y - z &= 1, \\ 2x - y + 2z &= 3.\end{aligned}$$

$$\begin{aligned}25. \quad 3x + 4y - 2z &= 0, \\ 2x + 3y - z &= 7.\end{aligned}$$

$$\begin{aligned}26. \quad 5x - 2y + z &= 3, \\ x + 2y - 2z &= 1.\end{aligned}$$

Solve the following for the ratio, $x:y:z$.

$$\begin{aligned}27. \quad x - 2y + 3z &= 0, \\ 3x + y - 2z &= 0.\end{aligned}$$

$$\begin{aligned}28. \quad x - y + z &= 0, \\ 3x - y + 2z &= 0.\end{aligned}$$

$$\begin{aligned}29. \quad x + y + z &= 0, \\ 2x - 3y - 6z &= 0.\end{aligned}$$

$$\begin{aligned}30. \quad 4x + y - 5z &= 0, \\ 3x - 2y - 2z &= 0.\end{aligned}$$

$$\begin{aligned}31. \quad 2x - y + 5z &= 0, \\ x + 4y - 2z &= 0.\end{aligned}$$

$$\begin{aligned}32. \quad 2x + y - 3z &= 0, \\ 4x - y + 8z &= 0.\end{aligned}$$

Chapter III

IMAGINARY NUMBERS

III-1. INTRODUCTION. Imaginary numbers are usually encountered first in algebra in the study of the quadratic equation. The student will recall that the formula for the solution of the quadratic equation, $ax^2 + bx + c = 0$, is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} . \quad [\text{III-1}]$$

The discriminant, $b^2 - 4ac$, may be negative, and it is this possibility that first brings the necessity of imaginary numbers to the student's attention.

There is need for further knowledge of imaginaries in more advanced topics in mathematics and in the sciences, especially in electrical theory. The purpose of this chapter is to advance the student's knowledge of imaginary numbers in general so that he may be better prepared to study their particular applications.

It is assumed in this chapter that the student is familiar with trigonometry and with the plotting of points on a system of rectangular coördinate axes, as well as with the introduction to imaginary numbers cited above. However, essential facts will be restated to call them to mind.

III-2. THE NUMBER i . By definition the square root of a number a is a number x such that x multiplied by itself equals a ; i.e.,

$$(x)(x) = x^2 = a.$$

As long as a is positive, there are two numbers, one positive and one negative, each of which multiplied by itself gives a . However, if a is negative, we are forced to the realization that we must introduce a new kind of number. The first step is to define x so that

$$x^2 = -1.$$

Definition. The square root of -1 is a number, denoted by the symbol $\sqrt{-1}$, such that

$$\sqrt{-1} \cdot \sqrt{-1} = -1.$$

The common notation for $\sqrt{-1}$ is i (except in electrical theory where i denotes current and j is used for $\sqrt{-1}$); i.e.,

$$i^2 = -1.$$

[III-II]

1 is called the unit of imaginaries.

We shall assume that real numbers combine with i in the same way that they combine with other literal factors in the operations of addition, subtraction, multiplication, and division.

$$\begin{array}{ll} \text{Examples: } 3i + 4i = 7i; & 5i - 2i = 3i; \\ (3i)(4i) = 12i^2 = -12; & \frac{18i}{6} = 3i. \end{array}$$

As a consequence of this assumption,

$$(-i)(-i) = i^2 = -1.$$

Hence there are two solutions of the equation $x^2 = -1$; viz.,

$$x = \pm i.$$

Extending the preceding definition one step further, we have the

Definition. The square root of $-a$, where a is positive, is i times the positive square root of a . In symbols

$$\sqrt{-a} = \sqrt{a}i.$$

$$\text{Hence, } \sqrt{-a} \cdot \sqrt{-a} = (\sqrt{a}i)(\sqrt{a}i) = ai^2 = -a;$$

$$\text{also, } (-\sqrt{-a})(-\sqrt{-a}) = (-\sqrt{a}i)(-\sqrt{a}i) = ai^2 = -a;$$

and, if a and b are both positive,

$$(\sqrt{-a})(\sqrt{-b}) = (\sqrt{a}i)(\sqrt{b}i) = \sqrt{ab}i^2 = -\sqrt{ab}.$$

(The definition excludes writing $\sqrt{-a} \cdot \sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$.)

It is convenient to have in mind a table of the powers of i .

$$i = i.$$

$$i^2 = i \cdot i = -1.$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i.$$

$$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = 1.$$

The higher powers of i continue in a cycle of four; $i^5 = i$, $i^6 = -1$, etc. $i^{k+4n} = i^k$; $k = 1, 2, 3, 4$; $n = 0, 1, 2, \dots$. In addition to the above table note $i^0 = 1$. The cycle for negative exponents will be left as a problem for the student. (See Problem 71, Exercise III-1.)

III-3. IMAGINARY NUMBERS. The quantity $a + bi$, where a and b are any real numbers, is called a complex number. a is called its real part; bi , its imaginary part.

If $b \neq 0$, $a + bi$ is called an imaginary number; if $b = 0$, $a + bi$ is simply a real number; if $a = 0$, $b \neq 0$, $a + bi$ is called a pure imaginary number.

Definition. Two imaginary numbers of the form $a + bi$ and $a - bi$ are called conjugate imaginary numbers. Each is said to be the conjugate of the other.

Definition. Two imaginary numbers are equal only if their real parts are equal and their imaginary parts are equal.

Example. $a + bi = a' + b'i$ if $a = a'$, $b = b'$.

Definition. The complex number zero is defined to be $0 + 0i$.

From the two preceding definitions it follows that $a + bi = 0$ only if $a = 0$, $b = 0$.

ADDITION AND SUBTRACTION OF IMAGINARY NUMBERS. The sum (or difference) of two imaginary numbers is the imaginary number obtained by adding (or subtracting) their real parts and their imaginary parts separately.

Examples.

$$(5 - 3i) + (6 + 7i) = 11 + 4i.$$

$$(8 - 2i) - (-10 + 3i) = 18 - 5i.$$

PRODUCT OF TWO IMAGINARY NUMBERS. The product of two imaginary numbers is found by multiplying them like two binomials. Reduce the result to the form of a single imaginary number.

Example.

$$\begin{aligned}(3 + 2i)(5 - 4i) &= 15 + (10 - 12)i - 8i^2 \\ &= 15 - 2i + 8 \\ &= 23 - 2i.\end{aligned}$$

QUOTIENT OF TWO IMAGINARY NUMBERS. The quotient of two imaginary numbers is found by multiplying both dividend (numerator) and divisor (denominator) by the conjugate of the divisor (denominator). Reduce the result to the form of a single imaginary number.

Example.

$$\begin{aligned}\frac{12 - 8i}{2 - 3i} &= \frac{(12 - 8i)(2 + 3i)}{(2 - 3i)(2 + 3i)} \\ &= \frac{24 + (36 - 16)i - 24i^2}{4 - 9i^2} \\ &= \frac{48 + 20i}{13} = \frac{48}{13} + \frac{20}{13}i.\end{aligned}$$

In division by complex numbers, as by real numbers, division by zero is excluded.

The four fundamental operations are summarized compactly as follows.

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i.$$

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i.$$

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

EXERCISE III-1

Write the following as pure imaginary numbers.

$$1. \sqrt{-25}. \quad 2. \sqrt{-3}. \quad 3. \sqrt{-16}. \quad 4. \sqrt{-50}. \quad 5. \sqrt{-128}.$$

Simplify the following.

$$6. -5i^3. \quad 7. 7i^{10}. \quad 8. \sqrt{-4}i^7. \quad 9. 100i^{100}. \quad 10. \sqrt{-49}i^{17}.$$

Solve the following equations, writing the answer in the form $a + bi$.

$$11. x^2 + 4 = 0. \quad 12. x^2 + 36 = 0. \quad 13. x^2 + 12 = 0.$$

$$14. x^2 + 20 = 0. \quad 15. x^2 + 48 = 0. \quad 16. x^2 + 63 = 0.$$

$$17. x^2 + x + 1 = 0. \quad 18. x^2 - 8x + 25 = 0.$$

$$19. x^2 + 6x + 34 = 0. \quad 20. x^2 + 3x + 5 = 0.$$

Carry out the indicated operations, and reduce answers to the form $a + bi$.

$$21. 5i - 7i + 6i. \quad 22. 10i - 2i - 3i + 7i.$$

$$23. (5i)(2i). \quad 24. (8i)(-7i).$$

$$25. (4i)(-9i). \quad 26. (-3i)(-11i).$$

$$27. \frac{24i}{8}. \quad 28. \frac{15i}{-5}. \quad 29. \frac{55i}{-11}. \quad 30. \frac{-60i}{-15}.$$

Reduce the following to the form $a + bi$.

$$31. (4 - 3i) - (2 + 5i). \quad 32. (8 - 5i) + (3 - 2i).$$

$$33. (11 + 6i) + (3 - 2i). \quad 34. (1 - i) - (2 + i).$$

$$35. (3 + 2i) - (5 + 2i).$$

$$36. (17 - 2i) + (8 + 7i) - (10 - 5i).$$

$$37. (7 - 4i) + (-3 - 6i).$$

$$38. (2 + i) - (4 - 3i) - (6 + 2i). \quad 39. (a + bi) + (a - bi).$$

$$40. (a + bi) - (a - bi). \quad 41. (4 - 5i)(2 + 3i).$$

42. $(8 - 31)(5 - 21)$.

43. $(11 + 21)(6 - 31)$.

44. $(2 + 1)(1 + 1)$.

45. $(3 - 51)(-2 - 41)$.

46. $(10 - 71)(-2 - 51)$.

47. $(7 + 31)(6 + 21)$.

48. $(3 - 21)(4 - 31)$.

49. $\frac{61}{21}$.

50. $\frac{61}{-31}$.

51. $\frac{-121}{41}$.

52. $\frac{-121}{-31}$.

53. $\frac{9 - 1}{4 - 51}$.

54. $\frac{4 + 191}{5 + 21}$.

55. $\frac{23 + 111}{7 - 1}$.

56. $\frac{7 + 1}{2 + 1}$.

57. $\frac{3 - 51}{4 + 21}$.

58. $\frac{6 - 1}{3 + 1}$.

59. $\frac{10 - 31}{5 - 21}$.

60. $\frac{8 + 91}{1 - 1}$.

61. $\frac{34}{5 + 31}$.

62. $\frac{40}{2 - 61}$.

63. $\frac{50}{7 + 1}$.

64. $\frac{73}{8 - 31}$.

65. $\frac{1}{1 + 1}$.

66. $\frac{1}{6 - 21}$.

67. $\frac{1}{1 - 1}$.

68. $\frac{1}{6 + 21}$.

69. Show that $\frac{1}{a + bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$.

70. Show that $\frac{1}{a + bi} = \frac{b}{a^2 + b^2} + \frac{a}{a^2 + b^2} i$.

71. Construct a table of negative powers of i : $i^{-1} = ?$; $i^{-2} = ?$; $i^{-3} = ?$; $i^{-4} = ?$; $i^{-k-4n} = ?$

III-4. THE COMPLEX PLANE. In the preceding sections we considered the fundamental operations on an exclusively algebraic basis; it is possible to carry out the same operations geometrically with valuable results.

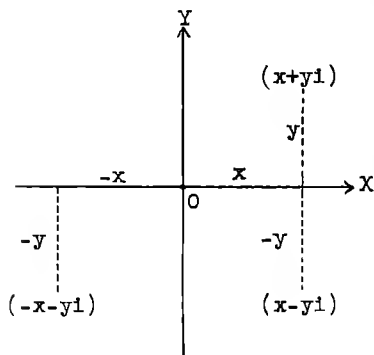


Fig. III-1

Let $x + yi$ be a typical imaginary number, the x and y corresponding to a and b , respectively, of the previous sections. Construct a set of rectangular coordinate axes, and plot the point (x, y) . This point is the geometric representation of the imaginary number $x + yi$. In this connection the x -axis is called the axis of reals, the y -axis is called the axis of imaginaries, and their plane is called the complex plane. Evidently a real number [$y = 0$] is plotted on the x -axis; and a pure imaginary number [$x = 0$] is plotted on the y -axis. In Fig. III-1 is shown the point $x + yi$, its conjugate $x - yi$, and its negative $-x - yi$.

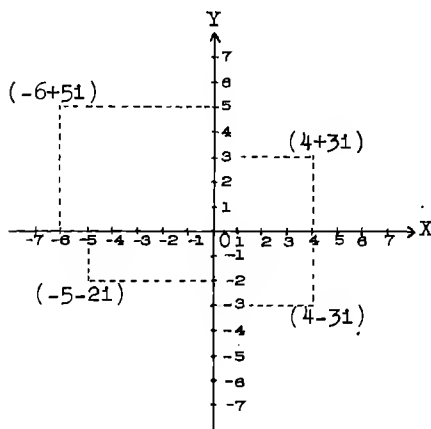


Fig. III-2

Fig. III-2 shows the positions of several points whose locations the student should verify.

Recall from §III-2 that the sum of $x_1 + y_1i$ and $x_2 + y_2i$ is $(x_1 + x_2) + (y_1 + y_2)i$. This operation is carried out geometrically in accordance with the following

Theorem. The sum of two imaginary numbers is represented by the fourth vertex of the parallelogram constructed on the lines joining the origin to the points which represent the given numbers as sides.

Proof. In Fig. III-3 let P_1 and P_2 be the points which represent $x_1 + y_1i$ and $x_2 + y_2i$, respectively, and construct a parallelogram on the lines OP_1 and OP_2 . Let P be the fourth vertex of the parallelogram.

Drop $ls P_2M_2$, P_1M_1 , and PM to the x -axis. Draw $P_1Q_1 \parallel$ to OX and intersecting PM at Q_1 . $\triangle P_1Q_1P = \triangle OM_2P_2$. ($OP_2 = P_1P$ by construction, and their \angle s are all equal because their sides are \parallel .) Hence $P_1Q_1 = OM_2$, $M_2P_2 = Q_1P$.

Line $OM = OM_1 + M_1M = OM_1 + P_1Q_1 = OM_1 + OM_2 = x_1 + x_2$.

Line $MP = MQ_1 + Q_1P = M_1P_1 + M_2P_2 = y_1 + y_2$.

Therefore, P represents $OM + MP = (x_1 + x_2) + (y_1 + y_2)i$.

If we identify the lines OP_1 and OP_2 as vectors [a vector is a quantity having magnitude and direction], then OP is the vector sum of OP_1 and OP_2 . In this form the construction is extremely important in physics where it is commonly called the parallelogram law.

Subtraction of one imaginary number from another is performed by the following

Rule. To subtract geometrically one imaginary number from another, replace the number to be subtracted by its negative, and apply the theorem for addition.

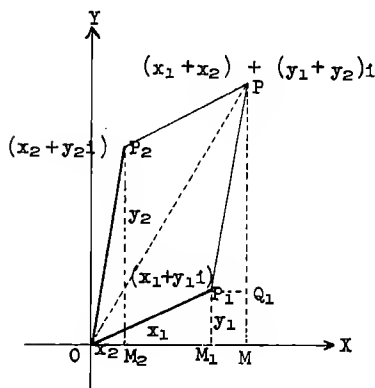


Fig. III-3

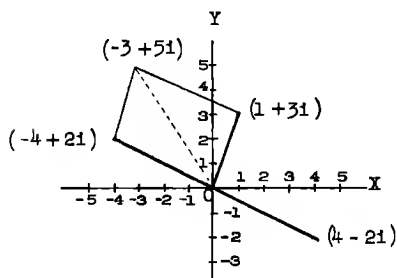


Fig. III-4

Fig. III-4 shows the details of

$$\begin{aligned}
 & (1 + 3i) - (4 - 2i) \\
 &= (1 + 3i) + (-4 + 2i) \\
 &= -3 + 5i.
 \end{aligned}$$

EXERCISE III-2

1. Plot the numbers (a) $4 + 2i$; (b) $4 - 3i$; (c) $-5 + i$; (d) $-4 - 5i$; (e) 6 ; (f) -1 ; (g) $4i$; (h) $-3i$.
2. Plot the numbers (a) $5 - 3i$; (b) $5 + 4i$; (c) $-2 + 6i$; (d) $-3 - i$; (e) $5i$; (f) 5 ; (g) $-4i$; (h) -7 .
3. (a) Plot the conjugate of each of the numbers in Problem 1.
(b) Plot the negative of each of the numbers in Problem 1.
4. (a) Plot the conjugate of each of the numbers in Problem 2.
(b) Plot the negative of each of the numbers in Problem 2.

Carry out the following operations geometrically.

5. $(3 - 2i) + (8 + 5i)$.
6. $(-3 - 2i) + (5 - 8i)$.
7. $(-2 + 6i) + (-3 + 4i)$.
8. $(5 + 2i) + (-3 - 3i)$.
9. $(-1 + 5i) + (4 - 7i) + (1 - 3i)$.
10. $(-2 - 4i) + (4 - 1) + (4 + 1)$.
11. $5 + (2 - 4i) + (-3 + 2i) + 3i$.
12. $-4i + (6 - 3i) + (1 + 5i) + 2$.
13. $(3 + 2i) - (1 - 6i)$.
14. $(4 - 3i) - (-2 + 5i)$.
15. $(-6 + 2i) - (-5 - 1)$.
16. $(6 + 3i) - (-1 + 4i)$.
17. $(-5 - 6i) - (3 + 3i)$.
18. $(3 - 5i) - (-2 - 2i)$.
19. $(4 + 3i) + 2i$.
20. $(4 + 3i) - 2i$.
21. $5 - (1 + 3i)$.
22. $5 + (1 - 3i)$.
23. $(5 + 6i) + (5 - 6i)$.
24. $(5 + 6i) - (5 - 6i)$.

25. $(-2 + 4i) + (-2 - 4i)$.

26. $(-2 + 4i) - (-2 - 4i)$.

27. Perform geometrically the operations in Problems 31 to 40 in Exercise III-1.

III-5. POLAR REPRESENTATION. The geometric representation of imaginary numbers presented in the preceding section is satisfactory for addition and subtraction, but is not well adapted to showing multiplication, division, raising numbers to powers (involution), or extracting roots (evolution). A different form, called the trigonometric or polar representation, is much better adapted to these operations.

Plot the number $x + yi$, and join the origin to it as shown in Fig. III-5. Denote the positive distance from the origin to the point by r , and the angle from the x -axis counter-clockwise to the line by θ . Then r and θ determine the point $x + yi$.

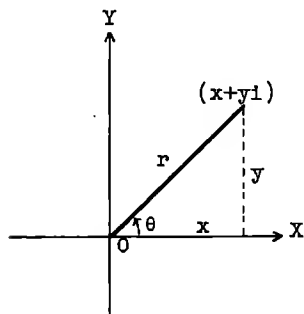


Fig. III-5

r is called the modulus, or absolute value of the number; θ is called its amplitude, or argument.

The representation by means of r and θ is called the polar, or trigonometric, representation of the number.

The relations between x, y and r, θ are found by trigonometry.

$$x = r \cos \theta,$$

$$y = r \sin \theta. \quad [\text{III-III}]$$

$$\begin{aligned} \text{Hence, } x + yi &= r \cos \theta \\ &+ (r \sin \theta)i = r(\cos \theta + i \sin \theta). \end{aligned}$$

Although r is always positive, this representation of the number is not unique since θ may denote any of the angles having the same initial and terminal sides as θ . These angles are all of the form $\theta + n \cdot 360^\circ$. The complete polar representation of the number $x + yi$ is $r[\cos(\theta + n \cdot 360^\circ) + i \sin(\theta + n \cdot 360^\circ)]$ where n is zero, or any positive or negative whole number. Unless there is some reason for doing otherwise, we shall consider only values of θ numerically less than 360° .

The formulas for r and θ in terms of x and y are

$$\left\{ \begin{aligned} \theta &= \tan^{-1} \frac{y}{x}; \quad \theta = \sin^{-1} \frac{y}{r}; \quad \theta = \cos^{-1} \frac{x}{r}; \text{ etc.} \\ \tan \theta &= \frac{y}{x}; \quad \sin \theta = \frac{y}{r}; \quad \cos \theta = \frac{x}{r}; \text{ etc.} \quad [\text{III-IV}] \\ r &= \sqrt{x^2 + y^2}. \end{aligned} \right.$$

$r = |x + yi|$ is an optional notation for the last of the above equations.

To change a number $x + yi$ to polar form, (1) compute $r = \sqrt{x^2 + y^2}$; (2) by inspection note in which quadrant $x + yi$ lies; (3) determine a suitable value of θ from $\tan \theta = \frac{y}{x}$ and the quadrant noted.

Example 1. Change $4 - 4i$ to polar form.

Solution. By inspection $x = 4$, $y = -4$, and $4 - 4i$ is in the fourth quadrant.

$$r = \sqrt{16 + 16} = 4\sqrt{2}; \quad \tan \theta = \frac{-4}{4} = -1, \quad \theta = 315^\circ.$$

Hence a polar form of $4 - 4i$ is

$$4\sqrt{2}(\cos 315^\circ + i \sin 315^\circ).$$

An equivalent form is $4\sqrt{2}[\cos(-45^\circ) + i \sin(-45^\circ)]$.

The complete polar representation is

$$4\sqrt{2}[\cos(315^\circ + n \cdot 360^\circ) + i \sin(315^\circ + n \cdot 360^\circ)].$$

When the angle θ cannot be determined by inspection, tables must be used. The three-place tables on page 83, COMPUTATION AND TRIGONOMETRY, are sufficient for the needs of this chapter.

Example 2. Change $-2 + 3i$ to polar form.

Solution. $x = -2$, $y = 3$, $-2 + 3i$ is in the second quadrant.

$$r = \sqrt{13}; \quad \tan \theta = -1.50; \quad \theta = 180^\circ - 56.3^\circ = 123.7^\circ.$$

The simplest polar form is $\sqrt{13}(\cos 123.7^\circ + i \sin 123.7^\circ)$.

The student may write other forms.

To change $r(\cos \theta + i \sin \theta)$ to the form $x + yi$, determine $\cos \theta$ and $\sin \theta$ by inspection or by tables, and multiply each term in the parenthesis by r .

Example 3. Change $10(\cos 240^\circ + i \sin 240^\circ)$ to the form $x + yi$.

$$\text{Solution.} \quad \cos 240^\circ = \frac{-1}{2}, \quad \sin 240^\circ = \frac{-\sqrt{3}}{2}$$

$$10(\cos 240^\circ + i \sin 240^\circ) = -5 - 5\sqrt{3}i$$

which is in the form $x + yi$.

III-6. MULTIPLICATION OF COMPLEX NUMBERS; DEMOIVRE'S THEOREM. Let two numbers in polar form be $r_1(\cos \theta_1 + i \sin \theta_1)$ and $r_2(\cos \theta_2 + i \sin \theta_2)$. Multiplying them out in full their product is

$$r_1 r_2 [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$$

Recognizing the trigonometric formulas

$$\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 = \cos(\theta_1 + \theta_2),$$

$$\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 = \sin(\theta_1 + \theta_2),$$

the product is

$$r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \quad [\text{III-V}]$$

Since the trigonometric formulas hold for all angles, [III-V] holds for all values of θ_1 and θ_2 .

This proves the

Theorem. The product of two imaginary numbers is an imaginary number whose modulus is the product of their moduli, and whose amplitude is the sum of their amplitudes.

Example 1. Multiply

$$[7(\cos 35^\circ + i \sin 35^\circ)][3(\cos 80^\circ + i \sin 80^\circ)].$$

Solution. Applying the theorem, the product is

$$21[\cos(35^\circ + 80^\circ) + i \sin(35^\circ + 80^\circ)] = 21(\cos 115^\circ + i \sin 115^\circ).$$

Example 2. Multiply

$$[8(\cos 120^\circ + i \sin 120^\circ)][4(\cos 40^\circ - i \sin 40^\circ)].$$

Solution. The second factor must first be put in precise polar form. By familiar trigonometric formulas,

$$4(\cos 40^\circ - i \sin 40^\circ) = 4(\cos 320^\circ + i \sin 320^\circ).$$

The theorem may now be applied, and the product is

$$32(\cos 440^\circ + i \sin 440^\circ) = 32(\cos 80^\circ + i \sin 80^\circ).$$

The product theorem may be extended to any number of factors in polar form. For example, the product

$$[r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)][r_3(\cos \theta_3 + i \sin \theta_3)] \\ = r_1 r_2 r_3 [\cos(\theta_1 + \theta_2 + \theta_3) + i \sin(\theta_1 + \theta_2 + \theta_3)]. \quad [\text{III-VI}]$$

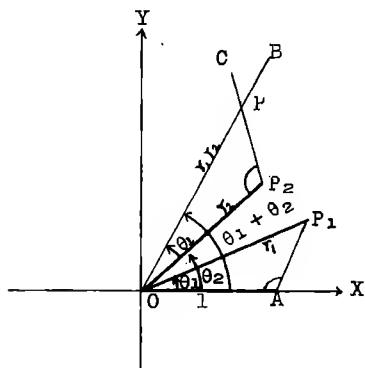


Fig. III-6

The multiplication of two numbers may be performed geometrically as follows. In Fig. III-6 let P_1 represent $r_1(\cos \theta_1 + i \sin \theta_1)$ and P_2 represent $r_2(\cos \theta_2 + i \sin \theta_2)$. Let OA denote the unit of length, and construct $\angle XOB = \theta_1 + \theta_2$. On line OP_2 construct $\angle OP_2C = \angle OAP_1$, and let P_2C and OB intersect at P .

The amplitude of P is $\theta_1 + \theta_2$ by construction. Also, by construction, $\triangle OP_2P$ is similar to $\triangle OAP_1$. Therefore,

$$\frac{OP}{r_2} = \frac{r_1}{1}; \text{ or, } OP = r_1 r_2.$$

Hence P represents the number $r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$, the product of the given numbers.

Raising a number to a positive, integral power is simply repeated multiplication of the number by itself. For example,

$$\begin{aligned} [r(\cos \theta + i \sin \theta)]^2 &= r^2 [\cos(\theta + \theta) + i \sin(\theta + \theta)] \\ &= r^2 (\cos 2\theta + i \sin 2\theta). \end{aligned}$$

By repeating this process we have the theorem known as

DeMoivre's Theorem. The n th power of a number $r(\cos \theta + i \sin \theta)$ is obtained by raising the modulus r to the n th power and multiplying the amplitude by n .

$$[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta) \quad [\text{III-VII}]$$

Example. Evaluate $[2(\cos 100^\circ + i \sin 100^\circ)]^5$.

Solution. By DeMoivre's Theorem

$$[2(\cos 100^\circ + i \sin 100^\circ)]^5 = 32(\cos 500^\circ + i \sin 500^\circ)$$

The amplitude may be simplified by diminishing it by 360° , so the answer may be written

$$32(\cos 140^\circ + i \sin 140^\circ). \quad \text{Ans.}$$

EXERCISE III-3

Change the following numbers to polar form and plot.

1. $5 + 5i$. 2. $-4\sqrt{3} - 4i$. 3. $-1 + \sqrt{3}i$. 4. $2 - 2i$.
5. $3 - 4i$. 6. $7 + 2i$. 7. $-2 + 5i$. 8. $-4 - 6i$.
9. $8i$. 10. $-8i$. 11. 5 . 12. -5 .
13. $-10i$. 14. $3i$. 15. -8 . 16. 6 .
17. $-3(\cos 45^\circ + i \sin 45^\circ)$. 18. $-4(\cos 140^\circ + i \sin 140^\circ)$.
19. $-10(\cos 240^\circ + i \sin 240^\circ)$. 20. $2(\cos 60^\circ - i \sin 60^\circ)$.
21. $5(\cos 150^\circ - i \sin 150^\circ)$. 22. $-8(\cos 330^\circ - i \sin 330^\circ)$.

Change the following numbers to the form $x + yi$ and plot

23. $2(\cos 45^\circ + i \sin 45^\circ)$. 24. $10(\cos 120^\circ + i \sin 120^\circ)$.
25. $8(\cos 300^\circ + i \sin 300^\circ)$. 26. $5(\cos 225^\circ + i \sin 225^\circ)$.
27. $4(\cos 25^\circ + i \sin 25^\circ)$. 28. $12(\cos 200^\circ + i \sin 200^\circ)$.
29. $6(\cos 180^\circ + i \sin 180^\circ)$. 30. $2(\cos 90^\circ + i \sin 90^\circ)$.
31. $7(\cos 0^\circ + i \sin 0^\circ)$. 32. $8(\cos 270^\circ + i \sin 270^\circ)$.
33. $10(\cos 360^\circ + i \sin 360^\circ)$. 34. $3(\cos 90^\circ + i \sin 90^\circ)$.

Find the following products leaving the answers in polar form. The operations may be performed geometrically when so directed by the instructor.

35. $[4(\cos 60^\circ + i \sin 60^\circ)] \cdot [3(\cos 150^\circ + i \sin 150^\circ)]$.
36. $[2(\cos 115^\circ + i \sin 115^\circ)] \cdot [7(\cos 185^\circ + i \sin 185^\circ)]$.
37. $[5(\cos 100^\circ + i \sin 100^\circ)] \cdot [3(\cos 170^\circ + i \sin 170^\circ)]$.
38. $(\cos 50^\circ + i \sin 50^\circ)[6(\cos 160^\circ + i \sin 160^\circ)]$.
39. $(\cos 30^\circ + i \sin 30^\circ)(\cos 60^\circ - i \sin 60^\circ)$.
40. $[20(\cos 200^\circ + i \sin 200^\circ)] \cdot [\frac{1}{2}(\cos 130^\circ + i \sin 130^\circ)]$.
41. $[16(\cos 110^\circ + i \sin 110^\circ)] \cdot [\frac{1}{4}(\cos 80^\circ - i \sin 80^\circ)]$.
42. $[6(\cos 70^\circ + i \sin 70^\circ)] \cdot [-3(\cos 20^\circ + i \sin 20^\circ)]$.
43. $[4(\cos 25^\circ + i \sin 25^\circ)] \cdot [-5(\cos 325^\circ + i \sin 325^\circ)]$.
44. $[3(\cos 105^\circ + i \sin 105^\circ)] \cdot [-4(\cos 75^\circ + i \sin 75^\circ)]$.
45. $[8(\cos 220^\circ + i \sin 220^\circ)] \cdot [3(\cos 70^\circ - i \sin 70^\circ)]$.
46. $[5(\cos 300^\circ + i \sin 300^\circ)] \cdot [-6(\cos 120^\circ - i \sin 120^\circ)]$.

Evaluate the following using DeMoivre's Theorem.

47. $[3(\cos 50^\circ + i \sin 50^\circ)]^3$.
 48. $[2(\cos 100^\circ + i \sin 100^\circ)]^6$.
 49. $(\cos 5^\circ + i \sin 5^\circ)^{10}$.
 50. $[4(\cos 200^\circ + i \sin 200^\circ)]^3$.
 51. $[5(\cos 80^\circ - i \sin 80^\circ)]^4$.
 52. $[2(\cos 135^\circ - i \sin 135^\circ)]^5$.
 53. $(\cos 18^\circ + i \sin 18^\circ)^5$.
 54. $[2(\cos 120^\circ + i \sin 120^\circ)]^3$.
55. Evaluate Problems 47 to 54 after replacing each exponent by its negative.

III-7. DIVISION OF COMPLEX NUMBERS; EVOLUTION. We shall next consider the division of one complex number by another using the polar form. Starting with the quotient

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$$

multiply numerator and denominator by $\cos \theta_2 - i \sin \theta_2$, using the polar form $\cos(-\theta_2) + i \sin(-\theta_2)$. Then

$$\begin{aligned} & \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \cdot \frac{\cos(-\theta_2) + i \sin(-\theta_2)}{\cos(-\theta_2) + i \sin(-\theta_2)} \\ &= \frac{r_1[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]}{r_2(\cos 0^\circ + i \sin 0^\circ)} \\ &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]. \quad [\text{III-VIII}] \end{aligned}$$

Theorem. The quotient of two numbers in polar form is a number whose modulus is the quotient of their moduli, and whose amplitude is the difference of their amplitudes.

$$\begin{aligned}\text{Example 1. } \frac{12(\cos 100^\circ + i \sin 100^\circ)}{2(\cos 70^\circ + i \sin 70^\circ)} \\ = 6(\cos 30^\circ + i \sin 30^\circ).\end{aligned}$$

To carry out geometrically the division of one number in polar form by another, let P_1 and P_2 represent $r_1(\cos \theta_1 + i \sin \theta_1)$ and $r_2(\cos \theta_2 + i \sin \theta_2)$, respectively, in Fig.

III-7. Let OA be the unit of distance.

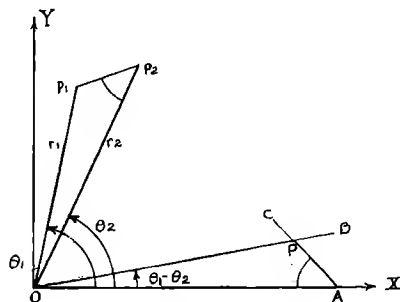


Fig. III-7

that are familiar. By factoring we solve the equation $x^2 - 1 = 0$, $[(x - 1)(x + 1) = 0, x = \pm 1]$ and say that ± 1 are the square roots of 1.

Similarly, factoring $x^3 - 1 = 0$, $(x - 1)(x^2 + x + 1) = 0$. Equating each factor to zero, and solving for x ,

$$x = 1, \frac{-1 \pm \sqrt{3}i}{2},$$

and these three values of x are the three cube roots of 1.

Again, factoring $x^4 - 1 = 0$, $(x - 1)(x + 1)(x^2 + 1) = 0$, and $x = 1, -1, \pm i$ are the four fourth roots of 1.

These examples suggest what is, in fact, the case; viz., that any number has n distinct n th roots.

We continue with a further example. Let it be required to find the fifth roots of $32(\cos 50^\circ + i \sin 50^\circ)$. We must find all the numbers such that each raised to the fifth power equals the given number. Let $r(\cos \theta + i \sin \theta)$ be such a number; then

$$\begin{aligned}[r(\cos \theta + i \sin \theta)]^5 &= 32(\cos 50^\circ + i \sin 50^\circ). \\ r^5(\cos 5\theta + i \sin 5\theta) &= 32(\cos 50^\circ + i \sin 50^\circ).\end{aligned}$$

Construct $\angle XO B = \theta_1 - \theta_2$, and $\angle OAC = \angle OP_2P_1$. Let OB and AC intersect at P . $\triangle OAP$ is similar to $\triangle OP_2P_1$. Then

$$\frac{OP}{r_1} = \frac{OA}{r_2}; \text{ or, } OP = \frac{r_1}{r_2}.$$

Therefore, P represents the quotient desired since its modulus is $\frac{r_1}{r_2}$, and its amplitude is $\theta_1 - \theta_2$.

Coming now to the extraction of roots of numbers, it will be well to review the facts

Equating moduli, $r^5 = 32$, $r = 2$; equating amplitudes, $5\theta = 50^\circ$, $\theta = 10^\circ$; and one fifth root is $2(\cos 10^\circ + i \sin 10^\circ)$.

To obtain others, recall that the complete polar form of the given number is $32[\cos(50^\circ + n \cdot 360^\circ) + i \sin(50^\circ + n \cdot 360^\circ)]$.

Hence, for $n = 1$, $5\theta = 50^\circ + 360^\circ$, $\theta = 82^\circ$;

for $n = 2$, $5\theta = 50^\circ + 720^\circ$, $\theta = 154^\circ$;

for $n = 3$, $5\theta = 50^\circ + 1080^\circ$, $\theta = 226^\circ$;

for $n = 4$, $5\theta = 50^\circ + 1440^\circ$, $\theta = 298^\circ$.

The remaining fifth roots are $2(\cos 82^\circ + i \sin 82^\circ)$, $2(\cos 154^\circ + i \sin 154^\circ)$, $2(\cos 226^\circ + i \sin 226^\circ)$, and $2(\cos 298^\circ + i \sin 298^\circ)$.

Any other permissible values of n , positive or negative, merely duplicate the roots already found.

The same reasoning may be carried out for the n th roots of any other number in polar form. Using general letters r , θ , and n leads to the following theorem.

Theorem. The n th roots of a number in polar form have as their common modulus the positive n th root of the modulus of the given number, and as their amplitudes,

$$\frac{\theta}{n}, \frac{\theta + 360^\circ}{n}, \frac{\theta + 2 \cdot 360^\circ}{n}, \dots, \frac{\theta + (n-1)360^\circ}{n}.$$

, [III-IX]

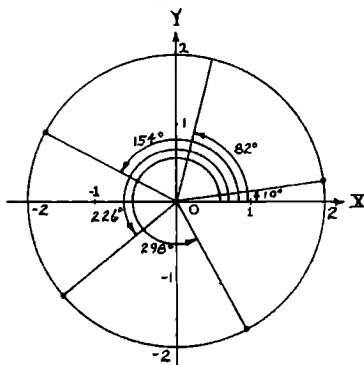


Fig. III-8

Geometrically the n th roots of a number all lie on the circumference of a circle with center at the origin, because they all have the same modulus. Since their amplitudes all differ by $\frac{360^\circ}{n}$, they are evenly spaced around the circumference. The fifth roots of $32(\cos 50^\circ + i \sin 50^\circ)$ are shown in Fig. III-8.

To find the n th roots of a number not in polar form, first express the number in polar form.

Example 2. Using the method of the theorem, verify that the cube roots of 1 are the same as those obtained by factoring.

Solution. In polar form. $1 = 1(\cos 0^\circ + i \sin 0^\circ)$.

The positive cube root of the modulus 1 is 1.

The amplitudes of the several roots are $\frac{0^\circ}{3}$, $\frac{0^\circ + 360^\circ}{3}$, and $\frac{0^\circ + 720^\circ}{3}$, which reduce to 0° , 120° , and 240° , respectively.

The polar forms of the cube roots of 1 are $1(\cos 0^\circ + i \sin 0^\circ)$, $1(\cos 120^\circ + i \sin 120^\circ)$, and $1(\cos 240^\circ + i \sin 240^\circ)$. Changing them to the form $x + yi$ they are 1, $\frac{-1 + \sqrt{3}i}{2}$, and $\frac{-1 - \sqrt{3}i}{2}$, respectively.

By combining the theorem on extracting roots with De-Moivre's Theorem we can evaluate any number k to a power $\frac{m}{n}$ where $\frac{m}{n}$ is rational. Let the desired operation be expressed as

$$k^{\frac{m}{n}} = [r(\cos \theta + i \sin \theta)]^{\frac{m}{n}}.$$

The possible n th roots are obtained, and each raised to the power m . The results are formulated as

$$\begin{aligned} & r^{\frac{m}{n}} \left(\cos \frac{m\theta}{n} + i \sin \frac{m\theta}{n} \right), \quad r^{\frac{m}{n}} \left[\cos \frac{m(\theta + 360^\circ)}{n} + i \sin \frac{m(\theta + 360^\circ)}{n} \right], \\ & \dots \dots \dots r^{\frac{m}{n}} \left[\cos \frac{m(\theta + (n-1)360^\circ)}{n} + i \sin \frac{m(\theta + (n-1)360^\circ)}{n} \right]. \end{aligned}$$

EXERCISE III-4

Evaluate the following quotients in polar form. Carry out the operation geometrically when so directed by the instructor.

1. $\frac{72(\cos 110^\circ + i \sin 110^\circ)}{8(\cos 55^\circ + i \sin 55^\circ)}$
2. $\frac{48(\cos 80^\circ + i \sin 80^\circ)}{16(\cos 60^\circ + i \sin 60^\circ)}$
3. $\frac{100(\cos 250^\circ + i \sin 250^\circ)}{25(\cos 175^\circ + i \sin 175^\circ)}$
4. $\frac{50(\cos 40^\circ + i \sin 40^\circ)}{10(\cos 70^\circ + i \sin 70^\circ)}$
5. $\frac{20(\cos 120^\circ + i \sin 120^\circ)}{5(\cos 30^\circ - i \sin 30^\circ)}$
6. $\frac{36(\cos 200^\circ - i \sin 200^\circ)}{9(\cos 100^\circ + i \sin 100^\circ)}$
7. $\frac{\cos 45^\circ + i \sin 45^\circ}{\cos 45^\circ - i \sin 45^\circ}$
8. $\frac{2(\cos 150^\circ - i \sin 150^\circ)}{(\cos 30^\circ - i \sin 30^\circ)}$
9. $\frac{24(\cos 330^\circ + i \sin 330^\circ)}{3(\cos 120^\circ - i \sin 120^\circ)}$
10. $\frac{28(\cos 225^\circ + i \sin 225^\circ)}{7(\cos 85^\circ + i \sin 85^\circ)}$
11. $\frac{4 - 4i}{-\sqrt{2} + \sqrt{6}i}$
12. $\frac{-3 + \sqrt{3}i}{1 + i}$
13. $\frac{-\sqrt{2} - \sqrt{2}i}{1}$
14. $\frac{40i}{5\sqrt{2} + 5\sqrt{2}i}$

In the following problems find the roots by operating on the number in polar form.

15. Find the fourth roots of 1, and verify the results obtained in the text by factoring.
16. Find the fourth roots of -1.
17. Find the cube roots of -1.
18. Find the fourth roots of 16i.
19. Find the cube roots of -8i.
20. Find the fifth roots of 32.
21. Find the fourth roots of $81(\cos 60^\circ + i \sin 60^\circ)$.
22. Find the cube roots of $125(\cos 135^\circ + i \sin 135^\circ)$.
23. Find the fifth roots of $(\cos 80^\circ + i \sin 80^\circ)$.
24. Find the sixth roots of $64(\cos 60^\circ + i \sin 60^\circ)$.
25. Find the cube roots of $4\sqrt{2} + 4\sqrt{2}i$.
26. Find the cube roots of $\frac{-27}{2} + \frac{27\sqrt{3}i}{2}$.
27. Find the fourth roots of $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.
28. Find the fourth roots of $-8 - 8\sqrt{3}i$.
29. Find the cube roots of $-8(\cos 120^\circ - i \sin 120^\circ)$.
30. Find the cube roots of $27(\cos 300^\circ - i \sin 300^\circ)$.
31. Find the fourth roots of $(\cos 240^\circ - i \sin 240^\circ)$.
32. Represent geometrically the results of any of the Problems 16 to 32 as directed by the instructor.
33. Evaluate $[8(\cos 150^\circ + i \sin 150^\circ)]^{\frac{2}{3}}$.
34. Evaluate $[16(\cos 300^\circ + i \sin 300^\circ)]^{\frac{3}{4}}$.

Chapter IV

SOLUTION OF EQUATIONS

IV-1. INTRODUCTION. The student is already familiar with methods of solving equations of the first degree (linear equations), and of the second degree (quadratic equations), by algebraic processes. It is possible to obtain algebraic solutions for equations of the third degree, and of the fourth degree, but the methods, while interesting, are complicated and not very practical for obtaining the solutions of numerical equations. It can be proved that the algebraic solution of the general equation of higher degree than the fourth is impossible. In the present chapter we shall develop instead of algebraic processes numerical methods of solving equations of higher degree than the quadratic. Some of these methods can be applied also to the solution of equations which are not algebraic. We begin with a brief review of the solution of the quadratic equation.

IV-2. SOLUTION OF THE QUADRATIC EQUATION BY FACTORING.

To solve a quadratic equation by factoring we write the equation so that the right member is zero and the left member is factored into linear factors. The argument then is that a product of factors is zero if and only if one or more of the factors is zero. Accordingly we set each of the factors containing the unknown equal to zero and find the values of the unknown that satisfy these equations.

Example. Solve $2x^2 - 5x = 12$.

Solution. $2x^2 - 5x - 12 = 0$

$$(2x + 3)(x - 4) = 0$$

$$2x + 3 = 0 \text{ or } x - 4 = 0$$

$$x = -\frac{3}{2}, \text{ or } x = 4.$$

When the factors can be readily found by inspection this method of solution is the easiest.

IV-3. SOLUTION OF THE QUADRATIC EQUATION BY COMPLETING THE SQUARE. We shall illustrate this method by examples.

Example 1. Solve $2x^2 - 5x = 12$.

Solution. We seek to make the left member a perfect square of the form

$$(1) \quad (x - p)^2 = x^2 - 2px + p^2.$$

By division we make the coefficient of x^2 equal 1,

$$x^2 - \frac{5}{2}x = 6.$$

Comparing with (1) we see that $p = \frac{1}{2}(\frac{5}{2}) = \frac{5}{4}$

Therefore we add $p^2 = (\frac{5}{4})^2 = \frac{25}{16}$ to both sides.

$$x^2 - \frac{5}{2}x + \frac{25}{16} = 6 + \frac{25}{16} = \frac{121}{16}$$

$$(x - \frac{5}{4})^2 = \frac{121}{16}, \quad x - \frac{5}{4} = \pm \frac{11}{4}$$

$$x = \frac{5}{4} \pm \frac{11}{4}, \quad x = 4 \text{ or } x = -\frac{3}{2}.$$

This method will give the roots also when they are not rational and therefore not readily found by factoring, as in the following:

Example 2. $3x^2 - 2x - 2 = 0$

Solution. $x^2 - \frac{2}{3}x = \frac{2}{3}$

$$x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$$

$$(x - \frac{1}{3})^2 = \frac{7}{9}$$

$$x - \frac{1}{3} = \pm \frac{\sqrt{7}}{3}, \quad x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}.$$

Example 3. Solve

$$(2) \quad ax^2 + bx + c = 0.$$

Solution. $ax^2 + bx = -c$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}, \quad \text{or}$$

$$(3) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

IV-4. SOLUTION OF THE QUADRATIC EQUATION BY FORMULA. The process of solving by completing the square explained in §IV-3 may be abbreviated by using the result of Example 3 without carrying out the whole process. For this purpose it is well to memorize equations (2) and (3) of that example. We illustrate the procedure:

Example. Solve by formula $5x^2 - 6x + 7 = 0$.

Solution. Comparing with (2), $a = 5$, $b = -6$, $c = 7$.
Then from (3)

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 5 \cdot 7}}{2 \cdot 5} \\ &= \frac{6 \pm \sqrt{36 - 140}}{10} \\ &= \frac{6 \pm 2\sqrt{-26}}{10} = \frac{3 \pm \sqrt{-26}}{5}. \end{aligned}$$

IV-5. FACTORS FROM ROOTS. In §IV-2 we saw that the solutions of a quadratic equation can be found when the left member can be factored and the right member is zero by setting each factor equal to zero. Evidently the reverse process may be used also and an equation having given roots can be made by forming the factors.

Example 1. Make an equation having the roots 2 and $-\frac{3}{5}$.

Solution. The factors corresponding to these roots are $x - 2$ and $x + \frac{3}{5}$, and the required equation is

$$(x - 2)(x + \frac{3}{5}) = 0 \quad \text{or} \quad 5x^2 - 7x - 6 = 0.$$

The process may be extended to factor second degree expressions which because of irrational or imaginary numbers cannot be factored by inspection.

Example 2. Factor

$$(1) \quad 3x^2 + x - 1.$$

Solution. Form the equation $3x^2 + x - 1 = 0$.

$$\begin{aligned} \text{Solving this by formula } x &= \frac{-1 \pm \sqrt{1 + 12}}{6} \\ &= -\frac{1}{6} \pm \frac{1}{6} \sqrt{13}. \end{aligned}$$

The factored equation then is

$$(2) \quad (x + \frac{1}{6} - \frac{1}{6}\sqrt{13})(x + \frac{1}{6} + \frac{1}{6}\sqrt{13}) = 0.$$

$$\text{Therefore } 3x^2 + x - 1 = 3\left(x + \frac{1}{6} - \frac{1}{6}\sqrt{13}\right)\left(x + \frac{1}{6} + \frac{1}{6}\sqrt{13}\right).$$

The factor 3 in this last product must be put in because inspection of the x^2 terms shows that (1) is 3 times the left member of (2).

IV-6. EQUATIONS IN QUADRATIC FORM. Some other quantity than x may be involved in an equation in a way similar to the x in the quadratic equations we have just considered. Such equations are said to be "in quadratic form."

Example 1. Solve $x^4 - 4x^2 - 5 = 0$.

Solution. Regarding x^2 as the unknown and solving by factoring,

$$(x^2 - 5)(x^2 + 1) = 0$$

$$x^2 - 5 = 0, x = \pm\sqrt{5}$$

$$x^2 + 1 = 0, x = \pm\sqrt{-1}.$$

Example 2. Solve $x^3 - \frac{8}{x^3} = 4$.

Solution. Clearing of fractions and reducing,

$$x^6 - 4x^3 - 8 = 0.$$

By formula, regarding x^3 as the unknown,

$$x^3 = \frac{4 \pm \sqrt{16 + 32}}{2} = 2 \pm 2\sqrt{3}$$

$$x = \sqrt[3]{2 + 2\sqrt{3}} \text{ or } x = \sqrt[3]{2 - 2\sqrt{3}}.$$

Example 3. Solve $e^{2x} - e^x = 2$.

Solution. Regarding e^x as the unknown,

$$e^{2x} - e^x - 2 = 0$$

$$(e^x - 2)(e^x + 1) = 0$$

$$e^x - 2 = 0, e^x = 2, x = \log_e 2 = 0.693$$

$$\text{or } e^x + 1 = 0, e^x = -1, \text{ impossible, reject.}$$

Example 4. Solve $1 + \cos x - 2 \sin^2 x = 0$.

Solution. By trigonometry, $\sin^2 x = 1 - \cos^2 x$. Hence the equation is

$$2 \cos^2 x + \cos x - 1 = 0.$$

Regarding $\cos x$ as the unknown and solving by formula,

$$\cos x = \frac{-1 \pm \sqrt{1 + 8}}{4} = \frac{-1 \pm 3}{4}$$

$$\cos x = \frac{1}{2} \text{ or } -1$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}, \text{ or } \pi, \text{ etc.}$$

EXERCISE IV-1

Solve the following:

1. $2x^2 - 13x - 24 = 0.$

2. $2x^2 - 13x + 23 = 0.$

3. $3x^2 - x - 1 = 0.$

4. $4x^2 - 12x + 9 = 0.$

5. $x^2 + 8x + 25 = 0.$

6. $35x^2 + 321x - 144 = 0.$

7. $12x^2 - 7x + 1 = 0.$

8. $5y^2 - y - 5 = 0.$

9. $70t^2 - 17t + 1 = 0.$

10. $w^2 - w + 1 = 0.$

Factor the following:

11. $105x^2 - 16x - 36.$

12. $24x^2 + 121x + 70.$

13. $2x^2 + 7x + 2.$

14. $3x^2 - 2x - 5.$

15. $x^2 + 1.$

16. $x^2 + x + 1.$

Solve the following:

17. $x^4 - 6x^2 + 8 = 0.$

18. $z^6 - 7z^3 - 8 = 0.$

19. $x^{1/4} + x^{1/2} = 12.$

20. $2x^2 - 3.811x + 1.434 = 0.$

21. $\pi x^2 - ex - 1.000 = 0.$

22. $100^x + 2(10^x) - 15 = 0.$

23. $(x^2 - 1)^2 - 7(x^2 - 1) + 12 = 0.$

24. $\frac{x^2 - 8}{x} + \frac{x}{x^2 - 8} = \frac{5}{2}.$ Hint: Let $\frac{x^2 - 8}{x} = z.$

25. $2x^4 - 3x^3 - x^2 - 3x + 2 = 0.$ Hint: Divide through by x^2 , add and subtract 4, and let $x + \frac{1}{x} = z.$

26. $ax^4 + bx^3 + cx^2 + bx + a = 0.$ Hint: Divide through by x^2 , add and subtract $2a$, and let $x + \frac{1}{x} = z.$

27. If $y = \sinh x = \frac{e^x - e^{-x}}{2}$ ($\sinh x$ is read "hyperbolic sine of x ") then $x = \sinh^{-1}y$ means "the inverse hyperbolic sine of y ", i.e., the result of solving for x in terms of y in the given equation. Get x in terms of y in a logarithmic form.

28. Given $y = \cosh x = \frac{e^x + e^{-x}}{2}$, get $x = \cosh^{-1}y$ in logarithmic form.

29. Solve for x , $8 \sin^4 x - 6 \sin^2 x + 1 = 0.$

30. If $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} = \frac{4}{3}$, get $\tan x.$

IV-7. THE RATIONAL, INTEGRAL EQUATION. We now consider methods of solution of certain equations of degree higher than the second, other than the special types treated in §IV-6. For the present we shall deal with the polynomial equation, also called the rational, integral equation, that is an equation which can be written in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0,$$

where n is a positive whole number, and a_0, a_1 , etc., are numerical coefficients.

We shall represent the left member of this equation by the symbol $f(x)$, (read "eff function of x ," or simply "eff of x ") throughout this chapter.

Once a particular meaning has been given to $f(x)$ in a special discussion or problem, with definite values of n and the a 's, it retains that meaning throughout that discussion or problem.

If x is replaced by another quantity in $f(x)$ it is similarly replaced in the expression for which $f(x)$ stands.

Example. If $f(x) = 2x^3 - 3x^2 + 4x - 1$

$$\text{then } f(y) = 2y^3 - 3y^2 + 4y - 1$$

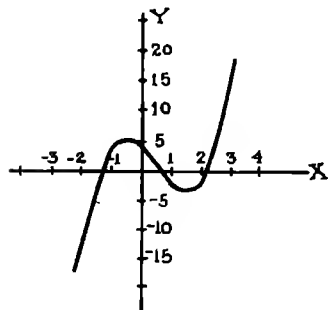
$$f(r) = 2r^3 - 3r^2 + 4r - 1$$

$$f(3) = 2 \cdot 3^3 - 3 \cdot 3^2 + 4 \cdot 3 - 1 = 38$$

$$\begin{aligned} f(-x) &= 2(-x)^3 - 3(-x)^2 + 4(-x) - 1 \\ &= -2x^3 - 3x^2 - 4x - 1. \end{aligned}$$

The student should distinguish between $f(x)$, the left member of the equation, the polynomial, and the equation $f(x)=0$.

IV-8. GRAPHICAL SOLUTION. We may find approximately the real roots of the equation $f(x) = 0$ by making a graph of $y = f(x)$. Intercepts of the curve on the x -axis give values of x that make $f(x) = 0$, i.e., are real roots of the equation.



Since the graph of $y = f(x)$ is continuous there is evidently at least one root between two values of x that make $f(x)$ have opposite signs.

Example. Solve graphically $2x^3 - 3x^2 - 5x + 4 = 0$.

Solution. We make a table of values

x	0	+1	+2	+3	-1	-2
y	+4	-2	-2	+16	+4	-14

Then we plot the graph using convenient scales for x and y . The graph shows that there is a root between -2 and -1 , one between 0 and $+1$, and one between $+2$ and $+3$. They might be estimated as about -1.4 , $+0.7$, $+2.2$.

Of course a more exact plot could be made by using fractional values of x .

IV-9. REMAINDER THEOREM AND FACTOR THEOREM. The graphical method just explained is fundamental for the methods of solution that we shall treat, but it is also helpful to have certain theorems dealing with the equation $f(x) = 0$. We shall state these theorems without proof. For proof the student may consult the instructor, or more extensive books on algebra.

REMAINDER THEOREM. If $f(x)$ be divided by $x - r$, the remainder is $f(r)$.

Example. Apply the Remainder Theorem to

$$f(x) = 2x^3 - 8x^2 + x + 7 \quad \text{with } r = 3.$$

Solution. By long division we have

$$\begin{aligned} f(x) \div (x - r) &= (2x^3 - 8x^2 + x + 7) \div (x - 3) \\ &= 2x^2 - 2x - 5 + \frac{-8}{x - 3}. \end{aligned}$$

That is the remainder is -8 . But $f(r) = f(3) = 2(3)^3 - 8(3)^2 + 3 + 7 = -8$.

A direct consequence of the Remainder Theorem is the

FACTOR THEOREM. If r is a root of the equation $f(x) = 0$, (i.e., $f(r) = 0$), then $x - r$ is a factor of the polynomial $f(x)$. Conversely: If $x - r$ is a factor of $f(x)$, r is a root of $f(x) = 0$.

The Factor Theorem and its converse have already been used in the special case of the quadratic. (§IV-5 and §IV-2.)

IV-10. SYNTHETIC DIVISION. The Remainder Theorem furnishes an alternative method of substituting values of x in $f(x)$ to form the table of values used in §IV-8. If we had to use ordinary long division, however, as in the Example of §IV-9, there would be little gain in using the theorem. But for the special kind of division here used, i.e., dividing a polynomial $f(x)$ by the divisor $x - r$, the division process can be shortened very much. We shall illustrate the method by examples.

The division used in the Example of §IV-9 written in full is

$$\begin{array}{r}
 2x^2 - 2x - 5 \longleftarrow \text{Quotient} \\
 2x^3 - 8x^2 + x + 7 \quad (x - 3 \text{ Divisor}) \\
 \underline{2x^3 - 6x^2} \\
 - 2x^2 + x \\
 \underline{- 2x^2 + 6x} \\
 - 5x + 7 \\
 \underline{- 5x + 15} \\
 -8 \text{ Remainder.}
 \end{array}$$

Let us agree to indicate exponents by the column in which the term stands, writing only coefficients, and let us drop all numbers which are always duplicates. The division then appears thus:

$$\begin{array}{r}
 \begin{array}{l} 2 \quad -8 \quad +1 \quad +7 \end{array} \quad (1 - 3 \text{ Divisor}) \\
 \begin{array}{r} \underline{-6} \\ -2 \end{array} \\
 \begin{array}{l} \text{Quotient} \quad \begin{array}{l} +6 \\ -5 \end{array} \end{array} \\
 \begin{array}{r} \underline{+15} \\ -8 \end{array} \text{ Remainder.}
 \end{array}$$

The arrangement is further condensed to occupy only three lines, and a further change is made in order to replace the subtraction steps by addition. This is done by changing the sign of the second term of the divisor. Also since the first term of the divisor always has the coefficient 1 it is unnecessary to write it. The above division is then finally reduced to this form:

$$\begin{array}{r}
 2 \quad -8 \quad +1 \quad +7 \quad (3 \text{ second term of divisor with sign changed,} \\
 \underline{+6 \quad -6 \quad -15} \quad \text{i.e., r.} \\
 2 \quad -2 \quad -5 \quad -8 \\
 \text{Quotient} \quad \text{Remainder}
 \end{array}$$

Of course in writing the result the x 's are supplied in the quotient, noting that the exponents in it are one less than in the same column in the dividend on account of the division by the linear factor. That is the quotient here is $2x^2 - 2x - 5$. The number written at the right (3 in this example) is r in the divisor $x - r$. Hence the remainder (-8 here) is the value of the dividend with that number substituted for x .

Example 2. Divide $x^4 - 3x^2 + 1$ by $x + 2$.

Solution. Note that since we are representing exponents by position missing terms must be supplied with zero coefficients.

$$\begin{array}{r}
 1 \quad +0 \quad -3 \quad +0 \quad +1 \quad (-2) \\
 \underline{-2 \quad +4 \quad -2 \quad +4} \\
 1 \quad -2 \quad +1 \quad -2 \quad +5
 \end{array}$$

i.e., $(x^4 - 3x^2 + 1) \div (x + 2) = x^3 - 2x^2 + x - 2 + \frac{5}{x + 2}$.

Example 3. Make the graph for solving the equation

$$x^4 - 3x^3 - 5x^2 + x + 7 = 0.$$

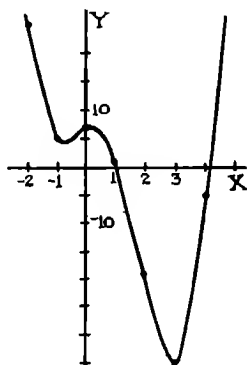
Solution. When $x = 0$, $y = f(0) = 7$.

To substitute $x = 1$ we divide by $x - 1$ and take the remainder. Similarly for other values of x . The work is

1	-3	-5	+1	+7	(1	
	+1	-2	-7	-6		
1	-2	-7	-6	+1		$f(1) = 1$
1	-3	-5	+1	+7	(2	
	+2	-2	-14	-26		
1	-1	-7	-13	-19		$f(2) = -19$
1	-3	-5	+1	+7	(3	
	+3	0	-15	-42		
1	0	-5	-14	-35		$f(3) = -35$
1	-3	-5	+1	+7	(4	
	+4	+4	-4	-12		
1	+1	-1	-3	-5		$f(4) = -5$
1	-3	-5	+1	+7	(5	
	+5	+10	+25	+130		
1	+2	+5	+26	+137		$f(5) = 137$
1	-3	-5	+1	+7	(-1	
	-1	+4	+1	-2		
1	-4	-1	+2	+5		$f(-1) = 5$
1	-3	-5	+1	+7	(-2	
	-2	+10	-10	+18		
1	-5	+5	-9	+25		$f(-2) = 25$

The table of values then is

x	-2	-1	0	+1	+2	+3	+4	+5
y	+25	+5	+7	+1	-19	-35	-5	+137



The graph is then as shown. There are roots at about $x = 1.1$ and $x = 4.0$.

IV-11. LIMITS FOR ROOTS. In making the table of values for the graph we have chosen values of x not very far from zero and substituted greater and greater values in the positive and negative directions. It is important to know how far the values must be carried. The following theorem gives limits outside of which no real roots can occur.

If successively greater positive values of x are substituted in $f(x)$ by synthetic division, $f(x) = 0$ can have no real

root greater than a value that makes all the signs in the last horizontal row in the division positive; and in substituting negative values of x in $f(x)$ by synthetic division $f(x) = 0$ can have no negative real root numerically greater than a value that makes the signs in the last row alternately plus and minus.

The truth of the theorem will be evident if we study the work in the last example for $x = 5$ and for $x = -2$. It is clear that a division using a numerically larger value of x would increase numerically every number in the last row. The sign of the remainder therefore cannot be changed by using a numerically larger value of x .

EXERCISE IV-2

1. If $f(x) = 4x^3 - 5x^2 - 6x + 7$, find $f(1)$, $f(-2)$, $f(y)$, $f(-x)$, $f(x+1)$.
2. If $f(x) = 2x^4 - 3x + 5$, find $f(2)$, $f(-1)$, $f(a)$, $f(-x)$, $f(\frac{1}{x})$.
3. Verify the Remainder Theorem if $f(x) = 2x^3 - 3x^2 + 5x - 3$ and $r = 4$. Perform the division both by long division and by synthetic division.
4. Verify the Remainder Theorem if $f(x) = 3x^4 - 2x^2 - 4x + 2$ and $r = -2$. Perform the division both by long division and by synthetic division.

Make the graph for solving each of the following equations. Substitute values of x by division, observing limits of the roots. Estimate the values of the real roots from the graph.

5. $x^3 - 3x^2 + x - 2 = 0$.
6. $2x^3 - 3x^2 - 12x + 1 = 0$.
7. $x^4 - 2x^3 + x^2 + 5x - 4 = 0$.
8. $2x^4 - 2x^3 + 7x - 3 = 0$.
9. $2x^4 - 13x^2 + 15 = 0$. Verify your roots by solving this also by §IV-6.
10. $3x^4 - 10x^2 + 8 = 0$. Verify your roots by solving this also by §IV-6.

IV-12. KINDS OF ROOTS. The student is already familiar with various kinds of numbers. First of all algebra uses positive and negative integers, for example, $+7$, -3 , then rational fractions, which are quotients of integers, like $\frac{2}{3}$, $-\frac{1}{5}$, 0.7 . Both these are included under the name of rational numbers. Then we have irrational numbers, which cannot be exactly expressed by either of the preceding, like $\sqrt{2}$, $-\sqrt[3]{7}$. All the kinds of numbers so far named are included in the class of real numbers. In addition there are imaginary numbers, also called complex numbers, such as $\sqrt{-1} = i$, $\sqrt{-3} = \sqrt{3}i$, $2 + 4\sqrt{-1} = 2 + 4i$, etc. These have been treated in Chapter III. As we have seen (§IV-4) such numbers sometimes occur in the solution of quadratic equations.

Rational integral equations may have numbers of any of these kinds for roots.

IV-13. FUNDAMENTAL THEOREM OF ALGEBRA. This theorem is as follows: Every rational, integral equation, $f(x) = 0$, has at least one root, real or complex.

Although the importance of this theorem is correctly indicated by its title the difficulty of its proof is such that it must be sought in advanced treatises, and we shall not give it.

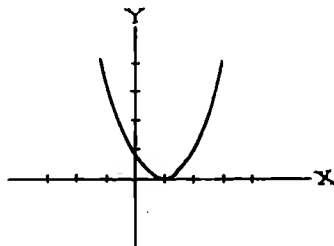
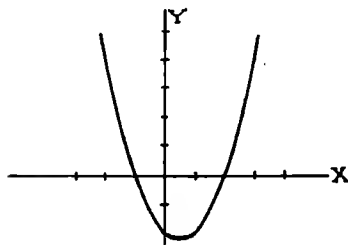
IV-14. NUMBER OF ROOTS. From the Fundamental Theorem and the Factor Theorem it follows at once that $f(x)$ contains at least one factor of the form $x - r$ (r either real or complex). If $f(x)$, of degree n , be divided by $x - r$ the quotient will be of degree $n - 1$. This quotient set equal to zero will constitute a new equation and the argument may be repeated until the quotient is free from x . It follows that $f(x)$ may be factored into n factors of the form $x - r$. (This is theoretically true. There may be practical difficulty in finding the factors.) If the factors are all different it is clear that $f(x) = 0$ has n roots. If some of the factors are alike we wish to have this statement still true. Therefore we agree to count each root of $f(x) = 0$ as many times as the corresponding factor of $f(x)$ occurs. With this understanding we state, the theorem: the rational integral equation of degree n has n roots, real or complex.

Example. Consider the equation $x^3 - 4x^2 + 5x - 2 = 0$.

Here direct substitution shows that $f(1) = 0$, that is, $x = 1$ is a root. Hence $x - 1$ is a factor, and we may write $f(x) = (x - 1)(x^2 - 3x + 2)$. Setting the quotient $= 0$, $x^2 - 3x + 2 = 0$. This has the root $x = 1$ again. Then $f(x) = (x - 1)(x - 1)(x - 2)$. We say that $x = 2$ is a single root, and $x = 1$ is a double root of $f(x) = 0$.

IV-15. GRAPH NEAR REPEATED ROOT. Let us next examine the shape of the graph of $y = f(x)$ near a repeated root of $f(x) = 0$. We show below several examples with the corresponding graphs, plotted in the usual way by means of a table of values.

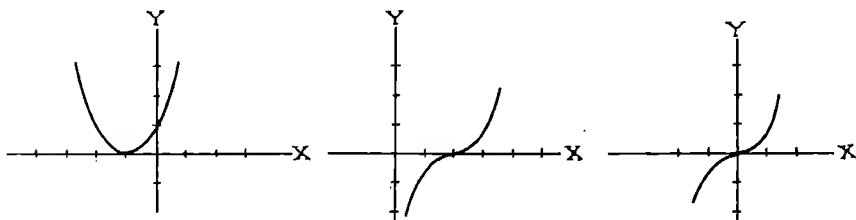
(a) $x^2 - x - 2 = (x+1)(x-2) = 0$. (b) $x^2 - 2x + 1 = (x-1)^2 = 0$.



(c) $(x + 1)^4 = 0$.

(d) $(x - 2)^3 = 0$.

(e) $x^5 = 0$.



We observe that if a real root occurs once, as for both roots in (a), the graph crosses the x -axis but is not tangent to it. If the root occurs twice or four times, i.e., an even number of times, as in (b) and (c), the curve is tangent to the x -axis at the root but does not cross it. If the root occurs an odd number of times more than once, as in (d) and (e), the curve crosses the axis and is tangent to it at the root.

It can be shown that these facts are true in general, and also that the graph behaves in a similar way near roots of any multiplicity when they occur with others in more complicated equations.

IV-16. COMPLEX ROOTS. We have already pointed out (§IV-12) that some of the roots of an equation $f(x) = 0$ may be complex numbers. We now state a theorem with regard to such roots.

In an equation $f(x) = 0$ in which the coefficients are real numbers, complex roots occur in conjugate pairs. That is, if $a + bi$ (a and b real numbers) is a root of such an equation, then $a - bi$ is also a root.

Formula (3) of §IV-3 for the roots of a quadratic equation shows the truth of the theorem for the quadratic. We shall omit the proof for equations of higher degree.

Example 1. Solve $x^4 - 7x^3 + 18x^2 - 7x - 13 = 0$, given that it has a root $3 + 2i$.

Solution. We divide out the factor corresponding to this root by synthetic division.

$$\begin{array}{r|rrrrr}
 1 & -7 & +18 & -7 & -13 & (3 + 2i) \\
 & 3 + 2i & -16 - 2i & 10 - 2i & +13 & \\
 \hline
 1 & -4 + 2i & 2 - 2i & 3 - 2i & 0 &
 \end{array}$$

By the theorem the equation also has a root $3 - 2i$, accordingly we divide out the corresponding factor from the quotient,

$$\begin{array}{r|rrrr}
 1 & -4 + 2i & 2 - 2i & 3 - 2i & (3 - 2i) \\
 & 3 - 2i & -3 + 2i & -3 + 2i & \\
 \hline
 1 & -1 & -1 & 0 &
 \end{array}$$

The zero remainder verifies the theorem. Since the quotient is now of the second degree the solution can be completed by setting the new quotient = 0 and solving the quadratic, thus:

$$x^2 - x - 1 = 0. \text{ By formula } x = \frac{1 \pm \sqrt{5}}{2}.$$

This problem may also be done in a slightly different way. As before we infer the presence of the conjugate complex root. The factors corresponding to the known roots then are

$$x - 3 - 2i \text{ and } x - 3 + 2i.$$

Since these are both factors of the given function their product is a factor. We form the product.

$$(x - 3 - 2i)(x - 3 + 2i) = x^2 - 6x + 13.$$

$f(x)$ may now be divided by this product (using long division, not synthetic division) obtaining $x^2 - x - 1$ for the quotient. The solution is then completed as before.

In this last solution we note that the product of a pair of factors corresponding to a pair of conjugate complex roots is a quadratic function with real coefficients. This is true in general. Combining this with the first theorem in this Article, and with the Factor Theorem, and §IV-14 we have the following:

A polynomial $f(x)$ with real coefficients can be factored into real linear or real quadratic factors, or both. (That is, the factors theoretically exist. There may be difficulty in finding them practically.)

In all ordinary cases the coefficients of $f(x)$ are real numbers. We shall assume in what follows, unless the contrary is stated, that such is the case.

Example 2. Form an equation of least possible degree with real coefficients among whose roots are 2, 1, $\frac{1}{2} - \frac{1}{2}\sqrt{3}i$.

Solution. For the coefficients to be real the conjugates of the complex roots must be present. Then

$$\begin{aligned} f(x) &= (x - 2)(x - 1)(x + 1)\left(x - \frac{1}{2} + \frac{1}{2}\sqrt{3}i\right)\left(x - \frac{1}{2} - \frac{1}{2}\sqrt{3}i\right) \\ &= (x - 2)(x^2 + 1)(x^2 - x + 1) \\ &= x^5 - 3x^4 + 4x^3 - 5x^2 + 3x - 2 = 0 \end{aligned}$$

is the required equation.

Since in plotting $y = f(x)$ we plot only points whose coordinates are real, complex roots of $f(x) = 0$ cannot be found from the graph. The presence of complex roots however may sometimes be inferred from the graph by means of the following theorem:

To each point on the graph of $y = f(x)$ where, as we trace continuously from left to right, the curve approaches the x -axis, then turns and recedes from it without reaching it, there corresponds a pair of complex roots.

The converse is not necessarily true.

Example 3. Let us plot the graph for solving $x^2 - 2x + 2 = 0$.

The curve is as shown.

The indication in the graph of the presence of a pair of complex roots is easily verified by solving the equation. Its roots are $x = 1 \pm i$.

Example 4. Let us plot the graph for solving $x^3 - 3x^2 + x - 3 = 0$.

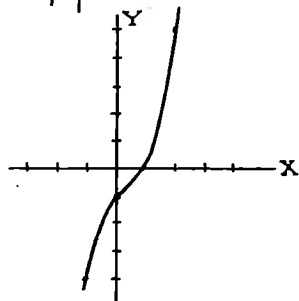
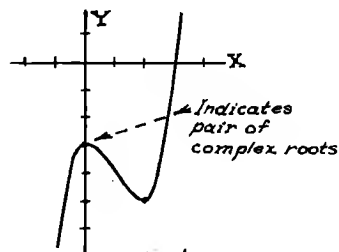
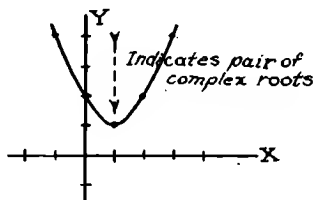
The curve is as shown.

The indication in the graph of the presence of a pair of complex roots is easily verified, for evidently it may be written $(x - 3)(x^2 + 1) = 0$ and the roots are $x = 3, \pm i$.

Example 5. Let us plot the graph for solving $x^3 - x^2 + x - 1 = 0$.

The curve is as shown.

There is no point in it indicating the presence of complex roots. But by factoring we have $(x - 1)(x^2 + 1) = 0$ and the roots are $x = 1, \pm i$. This illustrates the fact that the converse of the theorem is not necessarily true.



EXERCISE IV-3

Plot the graphs for solving the following equations, with special attention to the shape near the roots.

1. $(x^2 - x - 2)^2 = 0$.

2. $x^3 - 2x^2 = 0$.

3. $x^3 = 0$.

4. $(x^2 - 1)^2 = 0$.

5. $(x - 1)^4 = 0$.

6. $x(x - 1)^2(x + 2)^3 = 0$.

7. $(x + 2)^3(x + 1)(x - 2)^2 = 0$.

8. $(x + 3)^5(1 - x) = 0$.

Solve the following if one root is known as given:

9. $x^4 - 2x^3 + 3x^2 - 2x + 2 = 0$; $x = 1$.
 10. $x^4 - 2x^3 + 5x^2 - 8x + 4 = 0$; $x = 2i$.
 11. $2x^3 + 7x^2 + 16x + 15 = 0$; $x = -1 + 2i$.
 12. $3x^3 - 16x^2 + 31x - 20 = 0$; $x = 2 - i$.

Form an equation of least possible degree with real coefficients among whose roots are those given in the following:

13. 3, $\sqrt{2}i$. 14. -1, 1. 15. $\frac{1}{2}$, $2 + 3i$.
 16. $-\frac{2}{3}$, $1 - \sqrt{3}i$. 17. 1, $2 - i$. 18. $1 + 2i$, $5 - \sqrt{2}i$.
 19. 1, i , -2. 20. $1 + i$, $1 + i$.

IV-17. DESCARTES' RULE. Definition. A place in $f(x)$ where a positive coefficient is followed by a negative one, or a negative one is followed by a positive, is called a "change of sign." (It is understood that the terms are written in descending powers of x .)

DESCARTES' RULE. The equation $f(x) = 0$ has no more positive real roots than there are changes of sign in $f(x)$, and has no more negative real roots than there are changes of sign in $f(-x)$. We shall omit proof.

We call attention particularly to the fact that this rule does not give the number of positive and negative real roots. There may be less of each than the number of changes of sign in $f(x)$ and $f(-x)$.

Example 1. Apply Descartes' Rule to the equation

$$2x^3 - x^2 + x + 1 = 0.$$

Solution. There are two changes of sign in $f(x)$, hence there are not more than two positive roots.

$f(-x) = -2x^3 - x^2 - x + 1$ has one change of sign, hence the given equation has not more than one negative root. There are 3 roots in all (§IV-14) and so far as the Rule shows they may all be real. If there are less than three real roots the remaining roots must be complex. By §IV-16 there are either two or no complex roots in this equation.

Example 2. Apply Descartes' Rule to $x^4 - 3x^3 + 1 = 0$.

Solution. There are two changes of sign in $f(x)$, hence there are not more than two positive roots.

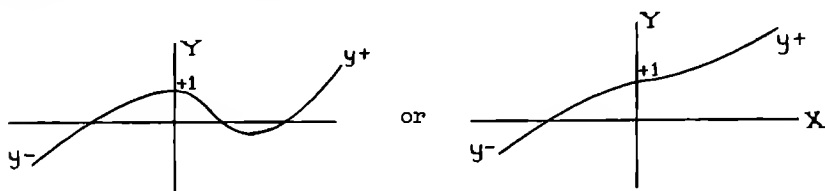
$f(-x) = x^4 + 3x^3 + 1$ has no changes of sign, hence there are no negative roots. There are 4 roots in all, hence the equation has surely 2 complex roots, and it may have 4 complex roots with no real ones at all.

IV-18. ROUGH GRAPHS. We can add to the information about roots given by Descartes' Rule by considering a rough graph. We substitute in $f(x)$ three values of x , namely, $-\infty$, 0 , $+\infty$, and note the corresponding signs of $y = f(x)$. In substituting ∞ (or $-\infty$), i.e., a numerically very large value of x , the sign of $f(x)$ will be that of the first term, a_0x^n , since by taking x large enough the numerical value of this term can be made larger than all others put together. The sign of $f(0)$ is of course that of the constant term, a_n . The possible crossings of the x -axis consistent with the signs of $f(x)$ for these three values frequently give useful information as to the roots. These results should be combined with those obtained by Descartes' Rule.

Example 1. Discuss by the rough graph $2x^3 - x^2 + x + 1 = 0$ (Example 1 of §IV-17).

Solution. $f(-\infty)$ has the same sign as $2(-\infty)^3$, i.e., $-$.

$f(0) = +1$, $f(+\infty)$ has the same sign as $2(+\infty)^3$, i.e., $+$. Combining these results with those previously obtained by Descartes' Rule the graph is either



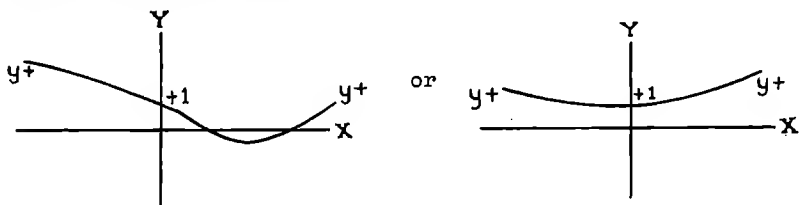
The possible combinations of roots may be conveniently put in tabular form, thus:

+	-	complex
2	1	0
0	1	2

Example 2. Discuss by rough graph $x^4 - 3x^3 + 1 = 0$. (Example 2 of the preceding Article.)

Solution. Here $f(-\infty)$ is $+$,
 $f(0)$ is $+$,
 $f(+\infty)$ is $+$.

Combining these results with those previously obtained by Descartes' Rule the graph is either



In tabular form the possibilities are	+	-	complex
	2	0	2
	0	0	4

Of course more exact location of the real roots can be found by making a complete numerical table of values as in §IV-8 but the methods of the last two Articles should be applied before making such a table.

EXERCISE IV-4

Tell all you can by Descartes' Rule and a rough graph about the roots of the following:

- | | |
|--------------------------------|--------------------------------|
| 1. $2x^3 + 6x^2 + 2x - 1 = 0.$ | 2. $3x^3 - 5x^2 - x + 4 = 0.$ |
| 3. $x^3 - 2x + 2 = 0.$ | 4. $x^3 + x^2 - 3 = 0.$ |
| 5. $2x^3 + x - 3 = 0.$ | 6. $5x^3 + x^2 + 3 = 0.$ |
| 7. $3x^4 - x^2 + 2x - 2 = 0.$ | 8. $4x^4 + x^3 + x^2 - 1 = 0.$ |
| 9. $2x^4 - x^2 - 3 = 0.$ | 10. $2x^4 + x^2 + 1 = 0.$ |
| 11. $x^5 - x^3 + x + 1 = 0.$ | 12. $4x^5 - x^2 + 2x - 2 = 0.$ |

In the four following problems n is a positive integer.

- | | |
|-------------------------|-------------------------|
| 13. $x^{2n} - 2 = 0.$ | 14. $x^{2n} + 2 = 0.$ |
| 15. $x^{2n+1} - 2 = 0.$ | 16. $x^{2n+1} + 2 = 0.$ |

IV-19. RATIONAL ROOTS. We have already pointed out (§IV-12) the possible kinds of roots an equation $f(x) = 0$ can have. We now give a theorem dealing with rational roots, i.e., roots which are either whole numbers, or the quotient of two whole numbers.

If the coefficients, $a_0, a_1, a_2, \dots, a_n$, in an equation $f(x) = 0$ are real rational integers, then any integral root is an exact divisor of a_n , and any root that is a rational fraction in lowest terms has a numerator that is an exact divisor of a_n and a denominator that is an exact divisor of a_0 . We shall omit proof.

An immediate consequence is that if an equation in which $a_0 = 1$ has a rational root it must be an integer.

Since a_0 and a_n can have only a finite number of exact divisors there are only a finite number of possible rational roots for an equation. These possibilities may be tested, substituting by synthetic division, and a finite number of trials will either find the rational roots or show that there are none. When a rational root has been found the quotient should be used as a new $f(x)$ for testing further possibilities.

When there is a large number of possible rational roots it frequently saves labor to locate the roots approximately by

the methods of §§IV-8, IV-17, and IV-18 first, then test possible rational roots that lie in intervals where the table of values or the graph indicates a root.

Example 1. Solve the equation $3x^4 - 7x^3 - 11x^2 + 21x + 10 = 0$ by first obtaining rational roots.

Solution. Possible rational roots are, ± 1 , ± 2 , ± 5 , ± 10 , $\pm \frac{1}{3}$, $\pm \frac{2}{3}$, $\pm \frac{5}{3}$, $\pm \frac{10}{3}$. To test all these 16 possibilities (if we did not happen to find a rational root sooner, which would shorten the process) would be laborious. Let us therefore make a table of values. $\begin{matrix} x & 0 & 1 \\ y & 10 & 16 \end{matrix}$. On substituting $x = 2$ by synthetic division we have

$$\begin{array}{r|rrrrr} 3 & -7 & -11 & +21 & +10 & (2 \\ & +6 & -2 & -26 & -10 & \\ \hline 3 & -1 & -13 & -5 & 0 & \end{array}$$

This shows $x = 2$ is a rational root. From this point on we use the equation obtained from the quotient, $3x^3 - x^2 - 13x - 5 = 0$.

This has possible rational roots -1 , ± 5 , $\pm \frac{1}{3}$, $\pm \frac{5}{3}$. ($+1$ is not possible, and therefore not listed here, since it was found not to be a root of the original equation.) The table of values for this equation is

$$\begin{matrix} x & 0 & +1 & +2 & +3 & -1 & -2 \\ y & -5 & -16 & -11 & +28 & +4 & -7 \end{matrix}$$

That the signs of y are opposite at $+2$ and $+3$, -1 and 0 , -2 and -1 , shows real roots between $+2$ and $+3$, between -1 and 0 , and between -2 and -1 , and there are no more roots. The only possible rational roots in these intervals are $-\frac{1}{3}$, and $-\frac{5}{3}$. We test these by synthetic division.

$$\begin{array}{r|rrrr} 3 & -1 & -13 & -5 & (-\frac{1}{3} \\ & -1 & +\frac{2}{3} & & \\ \hline 3 & -2 & & & \end{array}$$

The process need not be carried further, since evidently as soon as a fraction appears all later products will be fractions and the remainder cannot $= 0$.

$$\begin{array}{r|rrrr} 3 & -1 & -13 & -5 & (-\frac{5}{3} \\ & -5 & +10 & +5 & \\ \hline 3 & -6 & -3 & 0 & \end{array}$$

which shows $-\frac{5}{3}$ to be a root. The remaining factors are contained in the quotient, so we solve the equation

$$3x^2 - 6x - 3 = 0, \text{ or } x^2 - 2x - 1 = 0,$$

and thence by formula

$$x = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}.$$

The roots then are $2, -\frac{5}{3}, 1 \pm \sqrt{2}$.

Example 2. Examine for rational roots

$$4x^4 - 9x^3 + 23x^2 - 40x + 15 = 0.$$

Solution. According to the theorem of this article possible rational roots are $\pm 1, \pm 3, \pm 5, \pm 15, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{15}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{5}{4}, \pm \frac{15}{4}$. Descartes' Rule however shows that there are no negative roots so we test only the positive possibilities. The table of values is

x	0	+1	+2	+3
y	+15	-7	+19	+183

showing roots between 0 and 1, and between 1 and 2. We test the possible rational roots that are in these intervals.

4	-9	+23	-40	+15	$(\frac{1}{2})$
	+2	$-\frac{7}{2}$			
4	-7				
4	-9	+23	-40	+15	$(\frac{3}{2})$
	+6	$-\frac{9}{2}$			
4	-3				
4	-9	+23	-40	+15	$(\frac{1}{4})$
	+1	-2	$\frac{21}{4}$		
4	-8	+21			
4	-9	+23	-40	+15	$(\frac{3}{4})$
	+3	$-\frac{9}{2}$			
4	-6				
4	-9	+23	-40	+15	$(\frac{5}{4})$
	+5	-5	$\frac{45}{2}$		
4	-4	+18			

As these are the only possible rational roots in the intervals where there are roots we conclude that there are no rational

roots. The table of values indicates that there are two real irrational roots and two complex roots. This is not absolutely conclusive. There might be two more real roots, so close together that they do not appear from the table of values, and no complex roots, but this is very unlikely.

EXERCISE IV-5

In the following either solve as completely as possible by first finding rational roots or show that there are no rational roots.

1. $3x^3 + 13x^2 + 11x - 14 = 0$.
2. $3x^3 - x^2 + 4x + 4 = 0$.
3. $2x^3 - 11x^2 + 22x - 15 = 0$.
4. $3x^3 + 7x^2 + 19x - 65 = 0$.
5. $6x^3 - 15x^2 - 13x + 35 = 0$.
6. $4x^3 - 16x^2 - 24x + 15 = 0$.
7. $x^3 - 55x - 42 = 0$.
8. $x^3 + 3x^2 + x + 3 = 0$.
9. $2x^4 - x^3 - x^2 - x - 3 = 0$.
10. $12x^4 - 46x^3 + 36x^2 + 21x - 18 = 0$.
11. $x^4 - 2x^3 - 7x^2 + 9x + 12 = 0$.
12. $x^4 - 7x^3 + 5x^2 + 32x - 30 = 0$.
13. $24x^4 - 110x^3 + 125x^2 + 5x - 44 = 0$.
14. $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$.
15. $6x^4 + 22x^3 + 17x^2 - 4x + 4 = 0$.
16. $x^4 - 8x^3 + 19x^2 - 6x - 18 = 0$.
17. $x^5 - 3x^4 - 9x^3 + 31x^2 - 36 = 0$.
18. $x^5 - 5x^4 - 13x^3 + 65x^2 + 36x - 180 = 0$.
19. The equation $3x^5 + x^4 + 13x^3 + 2x^2 + 4x - 8 = 0$ has a complex root $2i$. It also has a rational root. Solve it completely.
20. The equation $6x^5 - 11x^4 + 59x^3 + 3x^2 + 20x - 50 = 0$ has a complex root $1 + 3i$. It also has a rational root. Solve it completely.

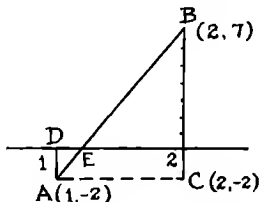
IV-20. IRRATIONAL ROOTS. We now show how irrational roots may be found to any required degree of accuracy. We shall explain the process by examples.

Example 1. Find an irrational root of $x^3 + 2x - 5 = 0$ correct to hundredths.

Solution. Descartes' Rule shows that there is not more than one positive root and that there are no negative roots. The rough graph shows that there is a positive real root. Evidently

this is neither 1 nor 5, the only possible rational roots. Therefore it is irrational. Form a table of values, x 0 1 2 .
 y -5 -2 7

This shows that the root is between 1 and 2. Now the idea is to find a value that approximately satisfies the equation, and we might next simply substitute fractions, as 1.5 or 1.1, etc., to locate the root, but it is better to choose the next value to try according to the values of y already found. Let us assume that in the interval from 1 to 2 the graph is a straight line and plot this interval to scale. If we rule the line AB and read the intercept from the graph it appears that the root is about 1.2. This result may also be obtained arithmetically without making the graph to scale. In the figure triangles ADE and ABC are similar. Therefore $\frac{DE}{AC} = \frac{AD}{CB}$, $DE = \frac{AD}{CB} \cdot AC$.



Hence $DE = \frac{2}{9} \cdot 1 = 0.2$, giving 1.2 as es-

timate of the root as before. This estimate must now be tested. We use synthetic division. Computations may be rounded off to the same number of decimal places as are required in the result, hundredths in this case.

$$\begin{array}{r|rrrr} 1 & 0 & +2 & -5 & (1.2 \\ & +1.2 & +1.44 & +4.13 & \\ \hline 1 & +1.2 & +3.44 & -0.87 & \end{array}$$

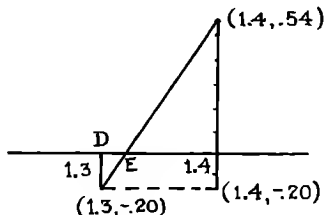
The sign of the remainder, -0.87, compared with the table of values, shows that the root is larger than 1.2. We try 1.3,

$$\begin{array}{r|rrrr} 1 & 0 & +2 & -5 & (1.3 \\ & +1.3 & +1.69 & +4.80 & \\ \hline 1 & +1.3 & +3.69 & -0.20 & \end{array}$$

The sign of the remainder shows that 1.3 is also smaller than the root. We try 1.4.

$$\begin{array}{r|rrrr} 1 & 0 & +2 & -5 & (1.4 \\ & +1.4 & +1.96 & +5.54 & \\ \hline 1 & +1.4 & +3.96 & +0.54 & \end{array}$$

This positive remainder shows that the root is less than 1.4. The first two figures of the root then are 1.3. To find the estimate for the next figure we proceed as before.



$$DE = \frac{0.20}{0.74} \cdot (0.1) = 0.027.$$

Therefore we estimate the root as $1.3 + 0.027 = 1.327 = 1.33$ to hundredths. We test this value

$$\begin{array}{r}
 1 \quad 0 \quad +2 \quad -5 \quad (1.33 \\
 \quad +1.33 \quad +1.78 \quad +5.03 \\
 \hline
 1 \quad +1.33 \quad +3.78 \quad +0.03
 \end{array}$$

This shows 1.33 too large. We test 1.32.

$$\begin{array}{r}
 1 \quad 0 \quad +2 \quad -5 \quad (1.32 \\
 \quad +1.32 \quad +1.74 \quad +4.94 \\
 \hline
 1 \quad +1.32 \quad +3.74 \quad -0.06
 \end{array}$$

This shows that the root lies between 1.32 and 1.33. Since the remainder for 1.33 is numerically less than that for 1.32 we infer that the root is nearer 1.33 than 1.32. Therefore we take $x = 1.33$ as the required root to hundredths.

Of course the process can be repeated further to find the root to a greater degree of accuracy if needed. In that case the later computations would need to be carried to more figures. The assumption on which the estimates are based, that within a short interval the curve is nearly a straight line, (in other words that the change in y is directly proportional to that in x), becomes more and more accurate at each step since each time the interval is only one tenth as large as it was before.

Example 2. Solve $3x^3 - 7x^2 - 8x - 1 = 0$ to hundredths.

Solution. Descartes' Rule and the rough graph show that the possible roots are + complex.

$$\begin{array}{ccc}
 1 & 0 & 2 \\
 1 & 2 & 0
 \end{array}$$

$$\begin{array}{cccccccc}
 \text{The table of values is } x & 0 & +1 & +2 & +3 & +4 & -1 & -2. \\
 y & -1 & -13 & -21 & -7 & +47 & -3 & -37
 \end{array}$$

This shows a root between 3 and 4. This is easily found by the method of Example 1 to be 3.20. We inquire now as to possible negative roots. That such inquiry is needed is shown by the graph. The dotted part of the curve is uncertain. It might or might not cross the x -axis between -1 and 0. Therefore we calculate $f(-0.5) = 0.825$. This shows two roots between -1 and 0. We locate these by substituting tenths by synthetic division. We omit the details of the work. We obtain

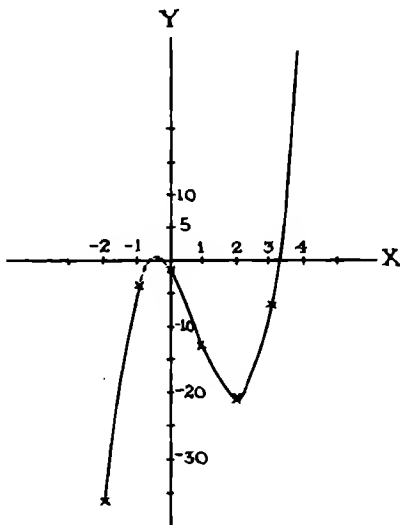
$$\begin{array}{cccccccc}
 x & -0.8 & -0.7 & -0.6 & -0.5 & -0.4 & -0.3 & -0.2 & -0.1 \\
 y & -0.616 & +0.141 & +0.632 & +0.825 & +0.888 & +0.689 & +0.296 & -0.273
 \end{array}$$

Thousandths are kept here because the quantities are so small. This shows a root between -0.8 and -0.7 and one between -0.2 and -0.1. We find the first of these. Its first figure is -0.7. Estimating the next figure by the method explained in Example 1 we have $\frac{0.141}{0.757} \cdot (0.1) = 0.02$ approximately, and the root is about -0.72. We test it by division

$$\begin{array}{rrrr}
 3 & -7 & -8 & -1 & (-0.72 \\
 & -2.16 & +6.595 & +1.012 & \\
 \hline
 3 & -9.16 & -1.405 & +0.012 &
 \end{array}$$

Comparing the sign of this remainder with the graph it appears that -0.72 is numerically too small, but the remainder is so small that it is safe to accept -0.72 as the root correct to hundredths without further testing.

In a similar way the other root is found to be -0.14 .



EXERCISE IV-6

Find the real roots of the following correct to three significant figures:

1. $x^3 - 8x - 19 = 0$.
2. $x^3 - 6x - 13 = 0$.
3. $x^3 + 5x - 13 = 0$.
4. $x^3 + 2x - 4 = 0$.
5. $x^3 - 2x^2 - x - 4 = 0$.
6. $x^3 + 3x^2 - 3x - 18 = 0$.
7. $x^3 + 3x^2 + x + 2 = 0$.
8. $x^3 + 6x^2 + 10x + 9 = 0$.
9. $x^4 + x^3 + 2x^2 - 3 = 0$.
10. $x^4 - 4x^3 + 4x^2 - x + 1 = 0$.
11. Find two roots of $3x^4 + 2x^3 - 18x^2 + 15 = 0$ between 1 and 2, correct to hundredths.
12. Find two roots of $3x^3 + 15x^2 + 16x - 7 = 0$ between -3 and -2 , correct to hundredths.

IV-21. SUMMARY. When we have no information as to the roots of an equation of degree higher than the second the complete procedure for finding the real roots is as follows:

1. Apply Descartes' Rule and the rough graph.
2. Make the table of values and sketch the complete graph as far as needed.
3. Note possible rational roots and test the ones that are in intervals where the graph indicates roots may lie. If

rational roots are found use the quotient of the division for the function thereafter.

4. Locate irrational roots from the graph and find them to the required accuracy by successive approximations.

EXERCISE IV-7

Find the real roots of the following, rational roots exactly, irrational roots to three significant figures. If all but two roots are rational solve completely.

1. $x^3 - 5x^2 - 4x - 36 = 0$.
2. $x^3 - x^2 - 5x - 2 = 0$.
3. $2x^3 - 2x^2 - 1 = 0$.
4. $2x^3 + 3x^2 + 1 = 0$.
5. $2x^4 + 7x^3 - 8x^2 - 25x - 6 = 0$.
6. $2x^4 - 7x^3 + 5x^2 + 6x - 36 = 0$.
7. $x^4 - 5x^2 - 4x - 36 = 0$.
8. $x^4 + x^2 - 5x - 20 = 0$.
9. $2x^4 + 5x^3 - 5x^2 - 13x + 7 = 0$.
10. $6x^4 + x^3 - 38x^2 - 6x - 36 = 0$.
11. $x^3 + 3x^2 - 14x + 1 = 0$.
12. $3x^3 - 5x^2 - 5x + 3 = 0$.
13. $2x^3 - x^2 - 8x - 2 = 0$.
14. $x^3 - 12x - 14 = 0$.
15. $x^3 - 1.33x^2 + 0.352 = 0$.
16. $x^3 + 2.73x^2 - 0.833 = 0$.
17. $3x^3 - 2.88x - 1.732 = 0$.
18. $2x^3 + 3.14x - 4.33 = 0$.
19. $x^3 - 12x^2 + 4x + 40 = 0$.
20. $x^3 - 3x^2 - 45x + 170 = 0$.
21. $x^4 - 2x^3 + x^2 + x - 1 = 0$.
22. $2x^4 + 5x^3 - 2x^2 + 2x - 3 = 0$.
23. $x^4 - 10x^2 - 14x - 2 = 0$.
24. $x^4 - 4x^3 + x^2 + 4x - 1 = 0$.
25. $x^4 - x^3 - 9x^2 + 10 = 0$.
26. $2x^4 - 8x^3 + 5 = 0$.
27. $4x^4 + x^3 + 12x^2 - 17x - 5 = 0$.
28. $6x^4 - 11x^3 + 19x^2 - 14x - 8 = 0$.
29. $4x^3 - 12x^2 + 8x - 1 = 0$.
30. $3x^3 - 12x^2 + 28 = 0$.
31. $x^4 - 4x^3 - 4x^2 + 16x + 3 = 0$.
32. $x^4 - 4x^3 - 4x^2 + 16x - 4 = 0$.
33. A box with open top is to be made from a rectangular piece of tin 16" long by 10" wide by cutting squares from the corners and turning up the sides. How large squares should be cut out if the box is to contain 144 cu. in.?
34. Find to hundredths two possible dimensions for the squares in the box of Problem 33 if the box is to contain 128 cu. in.

35. When an object floats in water the weight of the water displaced equals the weight of the object. The volume of a sphere is $\frac{4}{3}\pi R^3$ (R = radius), and the volume of a segment of a sphere is $\pi h^2(R - \frac{1}{3}h)$ (h = altitude of the segment). Specific gravity of an object is the ratio of the weight of the object to the weight of a like volume of water. Find to two decimals the depth at which a sphere of pine, specific gravity = 0.40, radius = 6.0" will float. (Call weight of water per cubic unit = w , this will divide out.)
36. A rectangular strip of carpet, 1.00 yd. wide, is to be laid diagonally on the floor of a room 4.00 yds. by 6.00 yds., each corner of the carpet just touching one side of the room. How far from the corner of the room will a corner of the carpet be, and how long must the carpet be? Solve to hundredths. (Use the former distance as the unknown, and get the latter from that afterwards.)

IV-22. TRANSCENDENTAL EQUATIONS. The graphical methods of this chapter may be applied to the solution of equations that are not algebraic, for instance those that contain trigonometric functions, logarithms, or exponentials, perhaps combined with algebraic terms. It must be understood that the theorems of the chapter do not apply to such equations, and substitution in such functions cannot be done by division.

ANSWERS

Page 4. Ex. I-1

- | | |
|-------------------------------------|------------------------------|
| 3. (a) 42; (b) 990; (c) 32,432,400 | |
| 5. 151,200 | 19. 336 |
| 7. 720 | 21. 2,494,800 |
| 9. 4096 | 23. 720 |
| 11. 1440 | 25. (a) 60; (b) 120; (c) 180 |
| 13. (a) 3600; (b) 4320 | 27. 13,699 |
| 15. (a) 4; (b) 8; (c) 32; (d) 2^n | 29. 5760 |
| 17. 5760 | 31. 672 |

Page 9. Ex. I-2

- | | |
|---|-----------------------------|
| 1. (a) 35; (b) 190; (c) 19,600; (d) 8; (e) 1; (f) 1 | |
| 3. 66 | 19. 2,598,960 |
| 5. 1140 | 21. (a) 1287; (b) 5148 |
| 7. 10,752 | 23. 34,594,560 |
| 9. (a) 10; (b) 15; (c) 31 | 25. (a) 495; (b) 19,958,400 |
| 11. 4 | 27. 31 |
| 13. 24,000 | 29. (a) 35; (b) 64 |
| 15. 302,400 | 31. 575,424 |
| 17. 1680 | |

Page 15. Ex. I-3

- | | |
|--|--|
| 1. (a) $\frac{1}{8}$; (b) $\frac{7}{8}$ | 3. (a) $\frac{14}{99}$; (b) $\frac{14}{33}$ |
|--|--|

5. $\frac{5}{16}$ 17. (a) $\frac{5}{16}$; (b) $\frac{1}{2}$
7. $\frac{1}{11}$ 19. $\frac{14}{285}$
9. $\frac{1}{12}$ 21. (a) $\frac{2}{15}$; (b) $\frac{8}{15}$; (c) $\frac{4}{5}$
11. $\frac{1}{10}$ 23. $\frac{2}{5}$
13. (a) Smith and Jones, $\frac{1}{8}$;
Brown, $\frac{1}{4}$; Robinson, $\frac{1}{2}$;
(b) $\frac{1}{4}$ for each
15. (a) $\frac{1}{3}$; (b) $\frac{1}{9}$; (c) $\frac{1}{9}$; (d) $\frac{1}{9}$ 25. $\frac{1}{90}$
27. $\frac{1}{5}$
29. $\frac{13}{25}$

Page 21. Ex. I-4

1. (a) $\frac{38569}{91914}$; (b) $\frac{2391}{91914}$ 9. (a) $\frac{21}{128}$; (b) $\frac{29}{128}$
3. $(\frac{26237}{89032})^2$ 11. (a) $\frac{135}{512}$; (b) $\frac{45}{512}$
5. $\frac{15222}{64563}$ 13. $\frac{64}{81}$
7. 0.01949 15. $\frac{360}{16807}$

Page 24. Ex. II-1

1. 25 9. 0 15. $x = \frac{1}{3}$, $y = -\frac{1}{2}$
3. 0 11. $k = -1$ 17. No solution
5. -10 13. $x = 4$, $y = 1$ 19. $x = 2$, $y = \frac{1}{7}$
7. -6

Page 27. Ex. II-2

1. -360 5. -12 9. $k = 2$, -10
3. -62 7. $k = \frac{7}{5}$ 13. $x = 2$, $y = 1$, $z = -1$

15. $x = 3, y = -1, z = 2$

17. $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{4}$

19. No solution

Page 38. Ex. II-3

1. 3

13. $x = 1, y = 3, u = -1$

3. -18

15. $x = 2, y = 1, z = -1, u = 3$

5. 3960

17. $x = 1, y = \frac{1}{2}, z = \frac{1}{3}, u = -1$

7. 55

19. $x = -1, y = -2, z = 1, u = -3$

9. -26

21. $x = 1, y = -1, z = 2, u = -2, w = 3$

11. 28

Page 43. Ex. II-4

1. $x:y:z = 3:2:1$

17. Inconsistent

3. Zero solution

19. $k = -5$

5. $x:y:z = 7:-5:2$

21. $k = 7, -19$

7. $x:y:z = 8:3:-1$

23. $x = 2z + 3, y = 1 - z$

9. $x:y:z = 1:-4:2$

25. $x = 2z - 28, y = 21 - z$

11. $x = \frac{3y}{2}, z = 0$

27. $x:y:z = 1:11:7$

13. $x = 3, y = -1$

29. $x:y:z = -3:8:-5$

15. $x = 2, y = 5$

31. $x:y:z = -2:1:1$

Page 48. Ex. III-1

1. 51

9. 100

17. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

3. 41

11. $\pm 2i$

19. $-3 \pm 5i$

5. $8\sqrt{2}i$

13. $\pm 2\sqrt{3}i$

21. 41

7. -7

15. $\pm 4\sqrt{3}i$

23. -10

25. 36	41. $23 + 21$	55. $3 + 21$
27. 31	43. $72 - 211$	57. $\frac{1}{10} - \frac{13}{10}1$
29. -51	45. $-26 - 21$	59. $\frac{56}{29} + \frac{5}{29}1$
31. $2 - 81$	47. $36 + 321$	61. $5 - 31$
33. $14 + 41$	49. 3	63. $7 - 1$
35. -2	51. -3	65. $\frac{1}{2} - \frac{1}{2}1$
37. $4 - 101$	53. $1 + 1$	67. $-\frac{1}{2} + \frac{1}{2}1$
39. 2a		

Page 55. Ex. III-3

1. $5\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$
3. $2(\cos 120^\circ + i \sin 120^\circ)$
5. $5(\cos 306.9^\circ + i \sin 306.9^\circ)$
7. $\sqrt{29}(\cos 111.8^\circ + i \sin 111.8^\circ)$
9. $8(\cos 90^\circ + i \sin 90^\circ)$
11. $5(\cos 0^\circ + i \sin 0^\circ)$
13. $10(\cos 270^\circ + i \sin 270^\circ)$
15. $8(\cos 180^\circ + i \sin 180^\circ)$
17. $3(\cos 225^\circ + i \sin 225^\circ)$
19. $10(\cos 60^\circ + i \sin 60^\circ)$
21. $5(\cos 210^\circ + i \sin 210^\circ)$
23. $\sqrt{2} + \sqrt{2}i$
25. $4 - 4\sqrt{3}i$
27. $3.62 + 1.69i$
29. -6
31. 7
33. 10
35. $12(\cos 210^\circ + i \sin 210^\circ)$
37. $15(\cos 270^\circ + i \sin 270^\circ)$
39. $\cos 330^\circ + i \sin 330^\circ$
41. $4(\cos 30^\circ + i \sin 30^\circ)$
43. $20(\cos 170^\circ + i \sin 170^\circ)$
45. $24(\cos 150^\circ + i \sin 150^\circ)$
47. $27(\cos 150^\circ + i \sin 150^\circ)$
49. $\cos 50^\circ + i \sin 50^\circ$
51. $625(\cos 40^\circ + i \sin 40^\circ)$
53. $\cos 90^\circ + i \sin 90^\circ$

$$55. \text{ From Prob. 47, } \frac{1}{27}(\cos 210^\circ + i \sin 210^\circ)$$

$$\text{From Prob. 49, } \cos 310^\circ + i \sin 310^\circ$$

$$\text{From Prob. 51, } \frac{1}{625}(\cos 320^\circ + i \sin 320^\circ)$$

$$\text{From Prob. 53, } \cos 270^\circ + i \sin 270^\circ$$

Page 59. Ex. III-4

$$1. 9(\cos 55^\circ + i \sin 55^\circ)$$

$$9. 8(\cos 90^\circ + i \sin 90^\circ)$$

$$3. 4(\cos 75^\circ + i \sin 75^\circ)$$

$$11. 2(\cos 195^\circ + i \sin 195^\circ)$$

$$5. 4(\cos 150^\circ + i \sin 150^\circ)$$

$$13. 2(\cos 135^\circ + i \sin 135^\circ)$$

$$7. \cos 90^\circ + i \sin 90^\circ$$

$$17. \cos 60^\circ + i \sin 60^\circ, \cos 180^\circ + i \sin 180^\circ, \\ \cos 300^\circ + i \sin 300^\circ$$

$$19. 2(\cos 90^\circ + i \sin 90^\circ), 2(\cos 210^\circ + i \sin 210^\circ), \\ 2(\cos 330^\circ + i \sin 330^\circ)$$

$$21. 3(\cos 15^\circ + i \sin 15^\circ), 3(\cos 105^\circ + i \sin 105^\circ), \\ 3(\cos 195^\circ + i \sin 195^\circ), 3(\cos 285^\circ + i \sin 285^\circ)$$

$$23. \cos 16^\circ + i \sin 16^\circ, \cos 88^\circ + i \sin 88^\circ, \\ \cos 160^\circ + i \sin 160^\circ, \cos 232^\circ + i \sin 232^\circ, \\ \cos 304^\circ + i \sin 304^\circ$$

$$25. 2(\cos 15^\circ + i \sin 15^\circ), 2(\cos 135^\circ + i \sin 135^\circ), \\ 2(\cos 255^\circ + i \sin 255^\circ)$$

$$27. \cos 15^\circ + i \sin 15^\circ, \cos 105^\circ + i \sin 105^\circ, \\ \cos 195^\circ + i \sin 195^\circ, \cos 285^\circ + i \sin 285^\circ$$

$$29. 2(\cos 20^\circ + i \sin 20^\circ), 2(\cos 140^\circ + i \sin 140^\circ), \\ 2(\cos 260^\circ + i \sin 260^\circ)$$

$$31. \cos 30^\circ + i \sin 30^\circ, \cos 120^\circ + i \sin 120^\circ, \\ \cos 210^\circ + i \sin 210^\circ, \cos 300^\circ + i \sin 300^\circ$$

$$33. 4(\cos 100^\circ + i \sin 100^\circ), 4(\cos 220^\circ + i \sin 220^\circ), \\ 4(\cos 340^\circ + i \sin 340^\circ)$$

Page 65. Ex. IV-1

1. $8, -\frac{3}{2}$ 3. $\frac{1}{6}(1 \pm \sqrt{13})$ 5. -4 ± 31
 7. $\frac{1}{4}, \frac{1}{3}$ 9. $\frac{1}{10}, \frac{1}{7}$ 11. $(3x - 2)(35x + 18)$
 13. $2(x + \frac{7}{4} - \frac{\sqrt{33}}{4})(x + \frac{7}{4} + \frac{\sqrt{33}}{4})$ 15. $(x + 1)(x - 1)$
 17. $\pm 2, \pm \sqrt{2}$ 19. 81 21. 1.1435, -0.2783
 23. $\pm 2, \pm \sqrt{5}$ 25. $\frac{1}{2}, 2, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$ 1
 27. $x = \log_6(y + \sqrt{y^2 + 1})$ 29. $n\pi \pm \frac{\pi}{6}, n\pi \pm \frac{\pi}{4}$

Page 70. Ex. IV-2

5. 2.9 7. 0.8, -1.4 9. $\pm 1.2, \pm 2.2$

Page 74. Ex. IV-3

9. $\pm 1, 1 \pm 1$ 11. $-1 \pm 21, -\frac{3}{2}$
 13. $x^3 - 3x^2 + 2x - 6 = 0$ 15. $2x^3 - 9x^2 + 30x - 13 = 0$
 17. $x^4 - 4x^3 + 6x^2 - 4x + 5 = 0$
 19. $x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2 = 0$

Page 80. Ex. IV-5

1. $\frac{2}{3}, -\frac{5}{2} \pm \frac{\sqrt{3}}{2}i$ 11. No rational roots
 3. $\frac{3}{2}, 2 \pm 1$ 13. $1, \frac{4}{3}, \frac{11}{4}, -\frac{1}{2}$
 5. No rational roots. 15. $-2, -2, \frac{1}{6} \pm \frac{\sqrt{5}}{6}i$
 7. $-7, \frac{7}{2} \pm \frac{1}{2}\sqrt{73}$ 17. 2, 2, 3, -3, -1
 9. $\frac{3}{2}, -1, \pm 1$ 19. $\pm 21, \frac{2}{3}, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$ 1

Page 83. Ex. IV-6

1. 3.64 3. 1.67 5. 2.85
 7. -2.89 9. 0.899, -1.17 11. 1.12, 1.81

Page 84. Ex. IV-7

- | | |
|--|-------------------------------|
| 1. 6.48 | 21. -0.755 |
| 3. 1.30 | 23. 3.73, -0.161 |
| 5. 2, $-\frac{3}{2}$, $-2 \pm \sqrt{3}$ | 25. 1.06, 3.40, -2.17, -1.28 |
| 7. 3.15, -2.84 | 27. $-\frac{1}{4}$, 1.15 |
| 9. $\frac{1}{2}$, 1.38 | 29. 0.162, 0.730, 2.11 |
| 11. 0.0726, 2.48, -5.55 | 31. 2.18, 3.93, -1.93, -0.181 |
| 13. 2.37, -1.605, -0.263 | 33. 2" |
| 15. -0.445 | 35. 5.2" |
| 17. 1.20 | 36. Length = 6.34 yds. |
| 19. 2.24, 11.3, -1.58 | |

Homework Sample

26.

$$2x^4 - 8x^3 + 5 = 0$$

$$\begin{array}{r} 2 \quad -8 \quad 0 \quad 0 \quad 5 \\ -4 \quad -8 \quad -16 \quad -27 \\ -2 \quad -6 \quad -18 \quad -49 \\ -12 \quad +24 \quad -48 \quad +101 \end{array}$$

$$\begin{array}{r} 2 \quad -8 \quad 0 \quad 0 \quad 5 \\ 1.6 \quad -5.12 \quad -0.96 \quad -0.2768 \\ -3.64 \quad -5.12 \quad -1.096 \quad +1.729 \end{array}$$

$$\begin{array}{r} 2 \quad -8 \quad 0 \quad 0 \quad 5 \\ 1.8 \quad -5.88 \quad -5.922 \quad -1.9198 \\ -2.62 \quad -5.58 \quad -5.922 \quad +4.81 \end{array}$$

x	y
0	5
1	-1
2	-27
3	-49
4	9
-1	15
-2	111
1.8	1.729
1.9	4.48
3.9	

$$x = .932$$

$$x = 3.946$$

Leon L

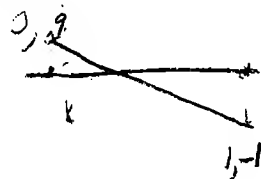
May 31, 1943

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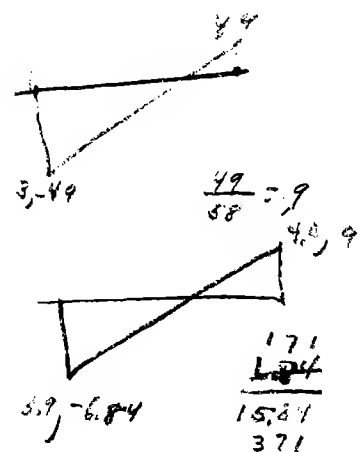
0.5

$$\begin{array}{r} 12 \\ 348 \\ \hline 418 \\ 37 \\ \hline = 0.32 \end{array}$$

$$\frac{5}{6} = .83$$



$$\begin{array}{r} 2 \quad -8 \quad 0 \quad 0 \quad 5 \\ 7.8 \quad -7.8 \quad -3.04 \quad -1.84 \\ -2 \quad -7.8 \quad -3.04 \quad -6.84 \end{array}$$



31.

$$x^4 - 4x^3 - 4x^2 + 16x + 3 = 0$$

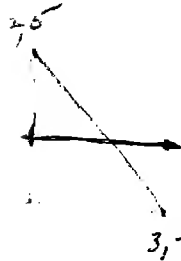
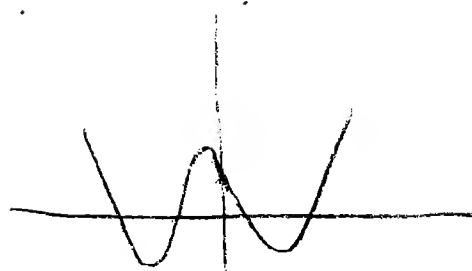
$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ -2 \quad -8 \quad 0 \quad 2 \\ \hline 1 \quad -7 \quad -5 \quad -12 \\ 0 \quad -4 \quad 0 \\ \hline 1 \quad -6 \quad +8 \quad 0 \quad 1 \\ -7 \quad +17 \quad - \quad - \end{array}$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ 2.3 \quad -3.91 \quad -18.2 \quad -5.06 \\ \hline 1 \quad -1.7 \quad -7.91 \quad -2.2 \quad -2.06 \end{array}$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ 2.2 \quad -3.96 \quad -1.55 \quad -2.3 \\ \hline 1 \quad -1.8 \quad -7.96 \quad -1.5 \quad -1.3 \end{array}$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ +2.1 \quad -3.99 \quad -1.68 \quad -1.68 \\ \hline 1 \quad -1.9 \quad -7.99 \quad -2 \quad +1.32 \end{array}$$

x	y
0	3
1	12
2	5
3	-12
4	3
-1	-12
-2	1
-3	+
2.3	-2.06
2.2	-3
2.1	1.32



$$\frac{5}{17} = .3$$

$$\frac{1.32}{1.62} \times 1 = .0825$$

$$x = 2.183$$

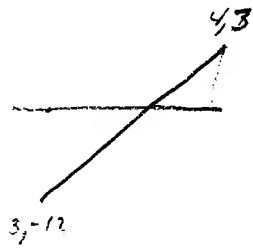
$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ 3.6 \quad -1.44 \quad -19.6 \quad -12.95 \\ \hline 1 \quad -4 \quad -5.44 \quad -3.6 \quad -9.95 \end{array}$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ 3.7 \quad -1.1 \quad -18.65 \quad -10.5 \\ \hline 1 \quad -3 \quad -5.1 \quad -2.85 \quad -7.5 \end{array}$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ 3.8 \quad -1.76 \quad -18.2 \quad -8.35 \\ \hline 1 \quad -2 \quad -4.76 \quad 2.2 \quad -5.35 \end{array}$$

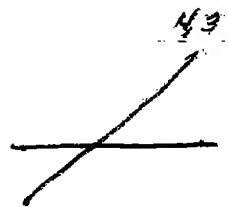
$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ 3.9 \quad -2.39 \quad -17.15 \quad -4.48 \\ \hline 1 \quad -1 \quad -4.39 \quad -1.15 \quad -1.48 \end{array}$$

x	y
3.6	-9.95
3.7	-10.5
3.8	-5.35
3.9	



$$\frac{12}{15} = .8$$

$$x = 3.93$$

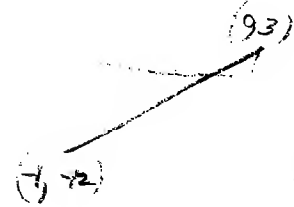


$$\frac{1.11}{4.98} \times 1 = .30$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ -1.2 \quad +.84 \quad +.63 \quad +3.32 \\ \hline 1 \quad -4.2 \quad -3.16 \quad +16.83 \quad +3.72 \end{array}$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ -1.1 \quad +.44 \quad +.36 \quad +1.64 \\ \hline 1 \quad -4.1 \quad -3.56 \quad +16.36 \quad +1.36 \end{array}$$

x	y
-1.2	-3.2
-1.1	1.36



$$\frac{3}{15} = .2$$

$$x = .181$$

$$\frac{68}{136} \times 1 = .081$$

$$\begin{array}{r} 1 \quad -4 \quad -4 \quad +16 \quad +3 \\ -1.9 \quad +1.12 \quad +13.08 \quad -4.4 \\ \hline 1 \quad -5.9 \quad +2.2 \quad +23.2 \quad -11 \end{array}$$

$$x = 1.94$$

$$\frac{1}{2.4} \times 1 = .04$$

$$\frac{11}{12} = .9$$